Key

Continuity

A graph is considered to be continuous when it is without breaks.

In order for a graph to be continuous at a point, 3 things must be true:

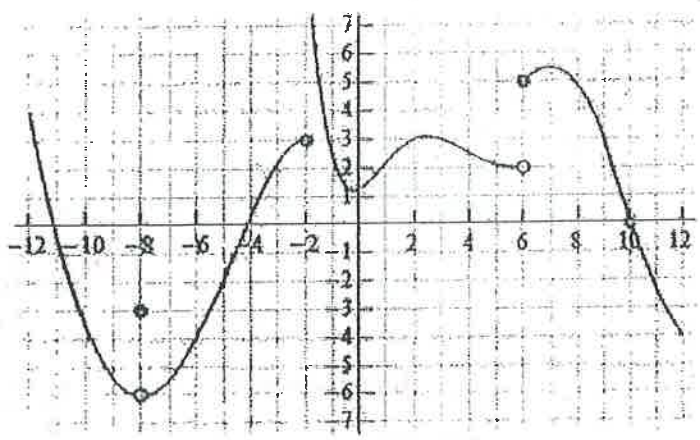
1. The value of the function must exist → f(x) exists
2. The limit of the function must exist → lim f(x) exists
3. The value of the function must equal the value of the limit → f(x) = lim f(x)

Finding discontinuity from a graph

When examining a graph, we look for points of discontinuity (when the function is not continuous or when the function has breaks.

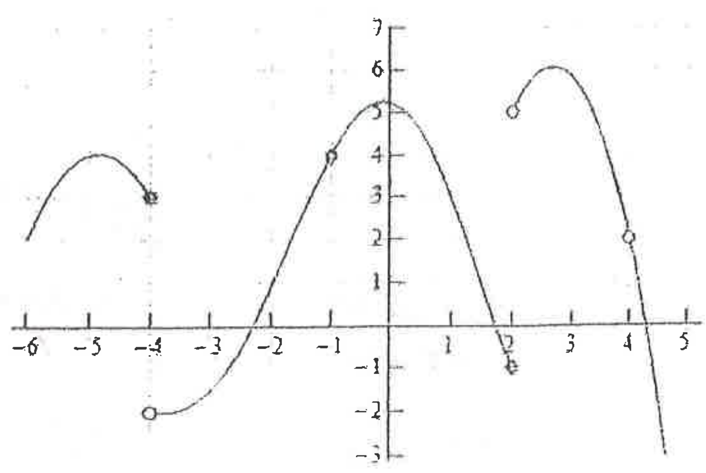
Example 1:
Determine where the graph is discontinuous.
Note why discontinuity occurs, using the three rules above.

x values	reason for discontinuity
-8	hole f(x) DNE
-2	asymptote $\lim_{x\to -2}$ DNE
6	hole $\lim_{x\to 6}$ DNE



Example 2:
Determine where the graph is discontinuous.
Note why discontinuity occurs, using the three rules above.

x values	reason for discontinuity
-4	lim DNE
-1	f(-1) DNE
2	lim DNE
4	f(4) DNE



Removing discontinuity

We can remove discontinuity when a function does not have an f(x) value but does have a limit.

To do so:

1. Determine the x-values where the function is discontinuous.
2. Attempt to find the limit at these x-values.
3. If the limit exists and is a numeric value, rewrite the function as a piecewise function using the limit value to fill in the hole.

Example 1:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$\lim_{x \to -1}$ $f(x) = \frac{x^2 + 3x + 2}{x + 1} = \frac{0}{0}$ $\frac{(x+1)(x+2)}{(x+1)} = x+2$

1. discontinuous at $x=-1$

$\lim_{x \to -1} x+2 = 1$

$f(x) = \begin{cases} \frac{x^2+3x+2}{x+1} & x \neq -1 \\ 1 & x = -1 \end{cases}$

Example 2:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$f(x) = \frac{x^2 + 3x + 2}{x - 1}$

1. f(x) is discontinuous at $x=1$

2. $\lim_{x \to 1} \frac{(x+1)(x+2)}{(x-1)} = \frac{6}{0}$

$\frac{6}{0}$ indicates an asymptote, not a hole

3. asymptote at $x=1$, discontinuity can't be removed

Example 3:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$f(x) = \frac{x^2 + x}{x}$

1. discontinuous at $x=0$

2. $\lim_{x \to 0} \frac{x^2+x}{x} = \frac{0}{0}$

$\frac{x(x+1)}{x} = x+1$

$\lim_{x \to 0} x+1 = 1$

3. $f(x) = \begin{cases} \frac{x^2+x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

Example 4:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$f(x) = \frac{x}{x^3}$

1. discontinuous at $x=0$

2. $\lim_{x \to 0} \frac{x}{x^3} = \frac{0}{0}$

$\frac{1}{x^2} = \frac{1}{0}$

indicates an asymptote, not a hole

3. asymptote at $x=0$, discontinuity cannot be removed

Example 5:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$f(x) = \frac{x^4 - 5x^2 + 4}{x^2 - 3x - 2}$

1. cannot factor

$x = \frac{3 \pm \sqrt{9 - 4(1)(-2)}}{2(1)} = \frac{3 \pm \sqrt{17}}{2}$

discontinuities at $x = \frac{3+\sqrt{17}}{2}$ and $x = \frac{3-\sqrt{17}}{2}$

2. $\lim_{x \to \frac{3+\sqrt{17}}{2}} \frac{x^4 - 5x^2 + 4}{x^2 - 3x - 2} = \frac{12\sqrt{17} + 52}{0}$

$\lim_{x \to \frac{3-\sqrt{17}}{2}} f(x) = \frac{52 - 12\sqrt{17}}{0}$

both are asymptotes, not holes

3. asymptotes at $x = \frac{3 \pm \sqrt{17}}{2}$, discontinuity cannot be removed

Intervals of continuity

When given a composition of functions (functions within functions) we can analyze both parts to determine the overall interval(s) of continuity.

To determine the interval(s) of continuity:

1. Identify f(x) and g(x) from the f∘g(x) function.
2. Determine where f(x) and g(x) are continuous.
3. If both parts are continuous everywhere, then the overall function is continuous everywhere.
4. If one part is continuous everywhere and the other part has a smaller interval, apply the smaller interval to the opposite function by writing an inequality.
5. Solve the inequality to find the interval where the overall function is continuous.

Example 1:

Determine the interval of continuity for the composition function.

$f(x) = e^{\sin x}$

1. $f(x) = e^u$ ← "u" indicates another function, not just x

$g(x) = \sin x$

2. e^u is continuous on $(-\infty, \infty)$

$\sin(x)$ is continuous on $(-\infty, \infty)$

3. ∴ $f(x) = e^{\sin x}$ is continuous on $(-\infty, \infty)$

Example 2:

Determine the interval of continuity for the composition function.

$f(x) = \ln(x^2 - 4)$

1. $\ln u$

$x^2 - 4$

2. $\ln u$ is continuous on $(0, \infty)$

$x^2 - 4$ is continuous on $(-\infty, \infty)$

3. N/A

4. $(0, \infty) = x > 0$

$x^2 - 4 > 0$

$x^2 > 4$

$x > 2$ or $x < -2$

5. $f(x) = \ln(x^2 - 4)$ is continuous on $(-\infty, -2) \cup (2, \infty)$

Example 3:

Determine the interval of continuity for the composition function.

$f(x) = \sqrt[3]{(3x + 6)}$

1. $f(x) = 3x + 6$

$g(x) = \sqrt[3]{u}$

2. $3x + 6$ is continuous on $(-\infty, \infty)$

$\sqrt[3]{u}$ is continuous on $(-\infty, \infty)$

3. ∴ $f(x) = \sqrt[3]{3x+6}$ is continuous on $(-\infty, \infty)$

Example 4:

Determine the interval of continuity for the composition function.

$f(x) = \sqrt{(2x)}$

1. $f(x) = 2x$

$g(x) = \sqrt{u}$

2. $2x$ is continuous on $(-\infty, \infty)$

$\sqrt{u}$ is continuous on $[0, \infty)$

3. N/A

4. $[0, \infty) = x \geq 0$

$2x \geq 0$

$x \geq 0$

5. $f(x) = \sqrt{2x}$ is continuous on $[0, \infty)$

Key

Finding Asymptotes of Rational Functions

Vertical Asymptotes:

1. Find any values of "x" that cause division by zero. These are likely locations of vertical asymptotes.
2. For all of the values found to cause division by zero find the limit of the function as "x" approaches those values.

   If "a" is one of those values find $\lim_{x \to a} f(x)$

   a. If the limit is finite then it is not an asymptote but rather a hole in function.

   b. If the limit diverges check whether it diverges negatively or positively on either side of the vertical asymptote.

   $\lim_{x \to a^+} f(x)$ The limit from the right of the asymptote.

   $\lim_{x \to a^-} f(x)$ The limit from the left of the asymptote.

Horizontal Asymptotes:

If either of the following limits exist (e.g. do not diverge) then there is a horizontal asymptote at $y = L_1$ and/or $y = L_2$

$$\lim_{x \to \infty} f(x) = L_1 \qquad \lim_{x \to -\infty} f(x) = L_2$$

Oblique Asymptotes:

1. Apply division of polynomials to the rational expression to break the rational expression into a sum of a polynomial part and a rational part: f(x) = P(x) + R(x).
2. Find the following limits:

$$\lim_{x \to \infty} R(x) \qquad \lim_{x \to -\infty} R(x)$$

3. If both of the limits are 0, we have an oblique asymptote of y = P(x).

Limits at Infinity/Finding Horizontal Asymptotes

In this chapter, we have been examining the first and second derivatives of a function in order to determine various features of the function's graph (relative and absolute extrema, intervals of increasing/decreasing, concavity, etc...). We will now use limits at infinity to examine the end behavior of functions and find any horizontal or slant asymptotes. We will then be reasonably equipped to sketch the graphs for a wide variety of functions.

Limits at Infinity

Examine the graph of the function $f(x)=\frac{3x^2}{x^2+1}$ with your graphing calculator.

Notice that as x increases or decreases without bound, the function $f(x)$ approaches 3.

This is denoted by

$\lim_{x\to\infty} f(x) = 3$ and $\lim_{x\to-\infty} f(x) = 3$

Notice that we are taking the limit of a function as x approaches infinity or negative infinity. These types of limits are called **limits at infinity**.

Our function approaches the line $y=$ 3 as x increases or decreases without bound. This line is called a horizontal asymptote of the graph of *f*.

Horizontal Asymptotes

The line $y=L$ is a horizontal asymptote of the graph of *f* if

$\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$

Strategy for finding Horizontal Asymptotes

Limits at infinity many of the same properties as the limits that we have discussed before. In addition, the following theorem is useful.

> If r is a positive rational number and c is any real number:
>
> 1. $\lim_{x\to\infty} \frac{c}{x^r} = 0$ and $\lim_{x\to-\infty} \frac{c}{x^r} = 0$

In general, if taking the infinite limit yields a constant in the numerator and infinity (or negative infinity) in the denominator, then the limit is 0.

Examples:

Find any horizontal asymptotes of the following functions:

a) $f(x) = 5 - \frac{2}{x^2}$ $\lim_{x\to\infty} 5 - \frac{2}{x^2} = 5$ $\lim_{x\to-\infty} 5 - \frac{2}{x^2} = 5$ horizontal asymptote $y=5$

b) $f(x) = \frac{2x+5}{3x^2+1}$ $\lim_{x\to\infty} \frac{x^2(\frac{2}{x}+\frac{5}{x^2})}{x^2(3+\frac{1}{x^2})} = \frac{0}{3} = 0$ $\lim_{x\to-\infty} f(x) = 0$ horizontal asymptote $y=0$

c) $f(x) = \frac{2x^2+5}{3x^2+1}$ $\lim_{x\to\infty} \frac{x^2(2+\frac{5}{x^2})}{x^2(3+\frac{1}{x^2})} = \frac{2}{3}$ $\lim_{x\to-\infty} f(x) = \frac{2}{3}$ horizontal asymptote $y=2/3$

d) $f(x) = \frac{2x^3+5}{3x^2+1}$ $\lim_{x\to\infty} \frac{x^2(2x+\frac{5}{x^2})}{x^2(3+\frac{1}{x^2})} = \infty$ $\lim_{x\to-\infty} f(x) = -\infty$ no horizontal asymptotes

Rational Functions

Many of the functions that you will be examining are rational functions. There are three possible cases, illustrated by the examples b, c, and d on the previous page. Rational functions will have the same limit at infinity and negative infinity.

Type A: If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.

Ex. $\lim_{x\to\pm\infty} \frac{2x+5}{3x^2+1} =$ Horizontal Asymptote: $y=0$

Type B: If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

Ex. $\lim_{x\to\pm\infty} \frac{2x^2+5}{3x^2+1} =$ Horizontal Asymptote: $y=2/3$

Type C: If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function does not exist (it is infinite).

Ex. $\lim_{x\to\pm\infty} \frac{2x^3+5}{3x^2+1} =$ Horizontal Asymptote: DNE

*This last type of rational function actually has a slant asymptote, which we will discuss later.

Example:

Find horizontal asymptotes for each of the following functions:

a) $f(x)=\frac{3-2x}{3x^3-1}$ h. asymptote $y=0$

b) $f(x)=\frac{3-2x}{3x-1}$ h. asymptote $y=-2/3$

c) $f(x)=\frac{3-2x^2}{3x-1}$ no horizontal asymptote

Slant Asymptotes

Let us examine a "Type C" case, where the infinite limit of a rational function does not exist (is infinite).

$$f(x)=\frac{2x^2-4x}{x+1}$$ $\frac{2x(x-2)}{x+1}$ doesn't cancel $\lim_{x\to\infty} f(x)=\infty$

This function has no horizontal asymptote, but it does have a *slant* asymptote. We can find it by using long division to rewrite the improper rational function.

$$\begin{array}{r} 2x-6+\frac{6}{x+1} \\ x+1\,\overline{)\,2x^2-4x+0} \\ -\,2x^2+2x \\ \hline -6x+0 \\ +6x+6 \\ \hline 6 \end{array}$$

$\lim_{x\to\infty} = 0$

$$\frac{2x^2-4x}{x+1}=2x-6+\frac{6}{x+1}$$

$\lim_{x\to\infty} = \infty$ $\lim_{x\to-\infty} = -\infty$

vertical asymptote @ $x=-1$

slant asymptote $y=2x-6$

As x approaches infinity or negative infinity, $\frac{6}{x+1}$ becomes very small (and thus insignificant) while $2x-6$ becomes very large (and thus very significant).

Therefore, as x increases or decreases without bound the function $f(x)=\frac{2x^2-4x}{x+1}$ looks very much like the function $g(x)=2x-6$ and $y=2x-6$ is slant asymptote of $f(x)$.

Use your calculator to sketch a graph of $f(x)$ to illustrate this concept.

Limits Approaching Infinity - Horizontal and Slant Asymptotes

1. $\lim_{x\to\infty}\frac{x^a}{x^b}$, *if* $a<b$. Then, limit = 0. Horizontal Asymptote at y = 0.

2. $\lim_{x\to\infty}\frac{Cx^a}{Dx^b}$, *if* $a=b$. Then, limit = $\frac{C}{D}$. Horizontal Asymptote at y = $^C/_D$.

3. $\lim_{x\to\infty}\frac{x^a}{x^b}$, *if* $a>b$. Then, limit = ∞. Limit = ∞ indicates that there is no horizontal asymptote.

Slant Asymptote: Find the equation through long division (the remainder is considered to be 0).

Limits Equal to Infinity - Vertical Asymptotes

1. f(a) does not exist but $\lim_{x\to a}f(x)$ = a constant value. "Hole" at x = a (no vertical asymptote).

2. f(a) does not exist and $\lim_{x\to a}f(x)$ = +∞, -∞, or DNE. There is a vertical asymptote at x = a.

For each equation:

a. Determine where the domain does not exist. At these points find the equation of the vertical asymptote or the coordinates of a "hole" in the function.

b. Find the limit approaching infinity. Use this calculation to write the equation of the horizontal asymptote or use long division to find the equation of the slant asymptote.

1. $f(x)=\frac{x^3-9x}{3x^2-6x-9}$ — $\frac{x(x+3)(x-3)}{3(x-3)(x+1)}$; hole @ x=3, v. asymptote @ x=−1; slant $y=\frac{1}{3}x+\frac{2}{3}$

2. $f(x)=\frac{x-4}{-4x-16}$ — $\frac{(x-4)}{-4(x+4)}$; hole @ x=4

3. $f(x)=\frac{1}{x^2+1}$ — horizontal y=0

4. $f(x)=\frac{5x^2}{10-3x^2}$ — horizontal $y=-\frac{5}{3}$; v. asymptote @ $x=\pm\frac{\sqrt{30}}{3}$

5. $f(x)=\frac{x^3-x^2-6x}{-3x^2-3x+18}$ — $\frac{x(x-3)(x+2)}{-3(x-2)(x+3)}$; v. asymptotes @ x=2, x=−3; slant $y=-\frac{1}{3}x+\frac{2}{3}$

6. $f(x)=\frac{x^3-16x}{-4x^2+4x+24}$ — $\frac{x(x-4)(x+4)}{-4(x+2)(x-3)}$; slant $y=-\frac{1}{4}x-\frac{1}{4}$; v. asymptotes @ x=−2, x=3

7. $f(x)=\frac{1}{3x^2+3x-18}$ — v. asymptotes @ x=2, x=−3; horizontal y=0

8. $f(x)=\frac{3x^2-12x}{x^2-2x-3}$ — $\frac{3x(x-4)}{(x-3)(x+1)}$; horizontal y=3; v. asymp. @ x=3, x=−1

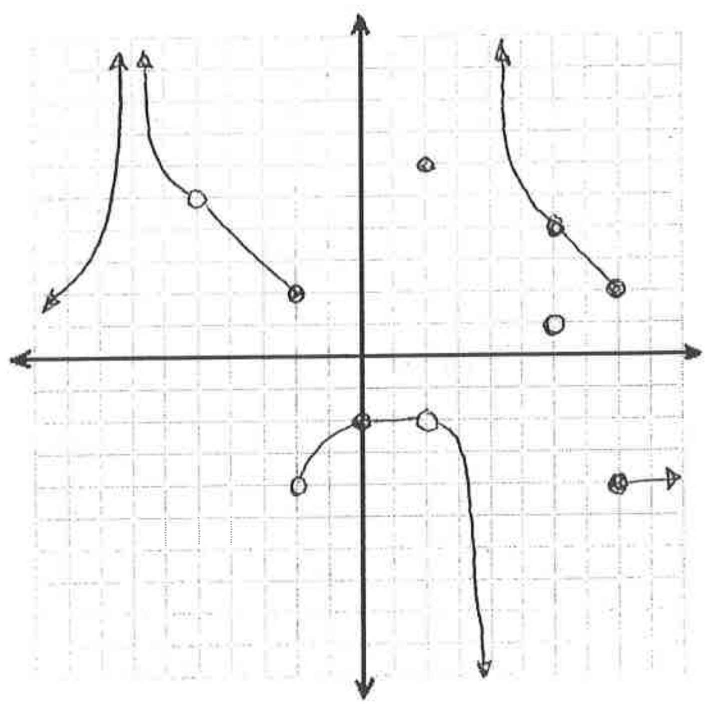
9. $f(x)=\frac{3}{x-2}$ — v. asymptote @ x=2; horizontal y=0

10. $f(x)=\frac{x^2+x-6}{-4x^2-16x-12}$ — $\frac{(x+3)(x-2)}{-4(x+1)(x+3)}$; hole @ x=−3; horizontal $y=-\frac{1}{4}$; v. asymptote @ x=−1

Continuity Assignment Name: ____________________

1. How is the concept of a limit related to the concept of continuity?

2. List all x values where the graph is discontinuous.
   Explain why discontinuity occurs at each location.



3. Would the graph of $x^2 + y^2 = 7$ be continuous at all points? Why or why not?

4. Give an example of a trig function where discontinuity can be removed.
   Provide a graph of this function.

For each function, list the intervals of continuity.

5. $f(x) = \sqrt[3]{x + 2}$

6. $f(x) = \sqrt{3x}$

7. $f(x) = \ln(4 - x^2)$

Note all points of discontinuity in each function.
Remove discontinuity when possible by creating a piece-wise function.
If discontinuity cannot be removed, explain why.

8. $f(x) = \dfrac{x^2 + 2x - 3}{x + 3}$

9. $f(x) = \dfrac{x^2}{x}$

10. $f(x) = \dfrac{3}{x^4}$

Using limits to identify asymptotes assignment

Name: ______________________________

For each function:

1. Identify any domain restrictions. At these restrictions, find the x value of any holes and /or the equations of any vertical asymptotes.
2. Use the concept of limits at infinity to determine whether there is a horizontal or slant asymptotes. Find the equation of this asymptote.

1. $f(x) = \dfrac{x^3 - x^2 - 6x}{-3x^2 - 3x + 18}$

2. $f(x) = \dfrac{x + 4}{-2x - 6}$

3. $f(x) = \dfrac{x^3 - 16x}{-4x^2 + 4x + 24}$

4. $f(x) = -\dfrac{4}{x^2 - 3x}$

Continuity

A graph is considered to be continuous when it is without breaks.

In order for a graph to be continuous at a point, 3 things must be true:

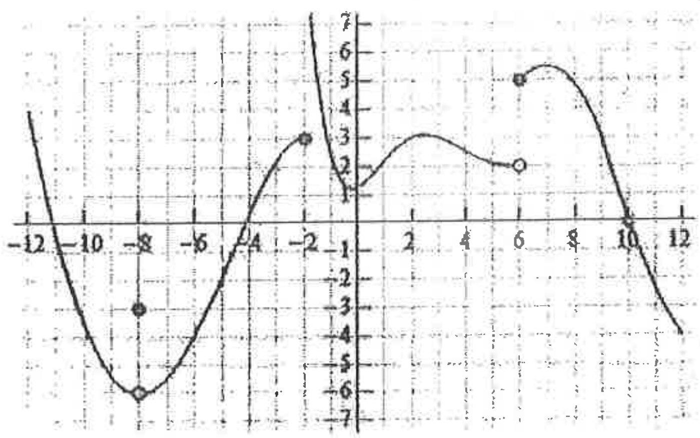
1. The value of the function must exist → f(x) exists
2. The limit of the function must exist → lim f(x) exists
3. The value of the function must equal the value of the limit → f(x) = lim f(x)

Finding discontinuity from a graph

When examining a graph, we look for points of discontinuity (when the function is not continuous or when the function has breaks.

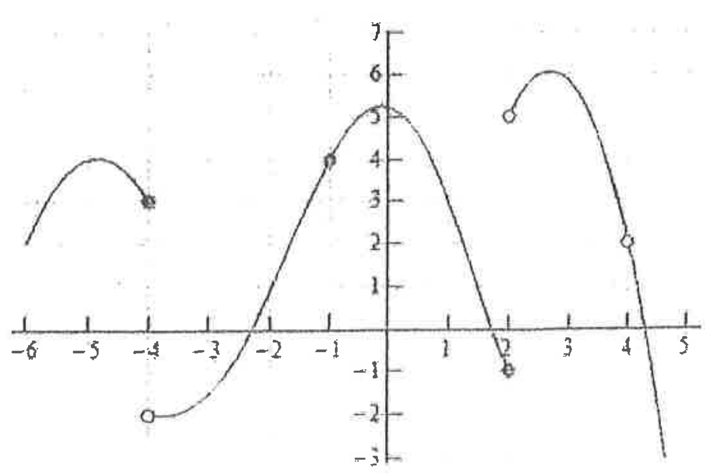
Example 1:
Determine where the graph is discontinuous.
Note why discontinuity occurs, using the three rules above.

x values	reason for discontinuity



Example 2:
Determine where the graph is discontinuous.
Note why discontinuity occurs, using the three rules above.

x values	reason for discontinuity



Removing discontinuity

We can remove discontinuity when a function does not have an f(x) value but does have a limit.

To do so:

1. Determine the x-values where the function is discontinuous.
2. Attempt to find the limit at these x-values.
3. If the limit exists and is a numeric value, rewrite the function as a piecewise function using the limit value to fill in the hole.

Example 1:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$$f(x) = \frac{x^2 + 3x + 2}{x + 1}$$

Example 2:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$$f(x) = \frac{x^2 + 3x + 2}{x - 1}$$

Example 3:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$$f(x) = \frac{x^2 + x}{x}$$

Example 4:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$$f(x) = \frac{x}{x^3}$$

Example 5:

Remove discontinuity when possible.
If discontinuity cannot be removed, explain why.

$$f(x) = \frac{x^4 - 5x^2 + 4}{x^2 - 3x - 2}$$

Intervals of continuity

When given a composition of functions (functions within functions) we can analyze both parts to determine the overall interval(s) of continuity.

To determine the interval(s) of continuity:

1. Identify f(x) and g(x) from the $f \circ g(x)$ function.
2. Determine where f(x) and g(x) are continuous.
3. If both parts are continuous everywhere, then the overall function is continuous everywhere.
4. If one part is continuous everywhere and the other part has a smaller interval, apply the smaller interval to the opposite function by writing an inequality.
5. Solve the inequality to find the interval where the overall function is continuous.

Example 1:

Determine the interval of continuity for the composition function.

$f(x) = e^{\sin x}$

Example 2:

Determine the interval of continuity for the composition function.

$f(x) = \ln(x^2 - 4)$

Example 3:

Determine the interval of continuity for the composition function.

$f(x) = \sqrt[3]{(3x + 6)}$

Example 4:

Determine the interval of continuity for the composition function.

$f(x) = \sqrt{(2x)}$

Finding Asymptotes of Rational Functions

Vertical Asymptotes:

1. Find any values of "x" that cause division by zero. These are likely locations of vertical asymptotes.

2. For all of the values found to cause division by zero find the limit of the function as "x" approaches those values.

   If "a" is one of those values find $\lim_{x \to a} f(x)$

   a. If the limit is finite then it is not an asymptote but rather a hole in function.

   b. If the limit diverges check whether it diverges negatively or positively on either side of the vertical asymptote.

   $\lim_{x \to a^+} f(x)$ The limit from the right of the asymptote.

   $\lim_{x \to a^-} f(x)$ The limit from the left of the asymptote.

Horizontal Asymptotes:

If either of the following limits exist (e.g. do not diverge) then there is a horizontal asymptote at $y = L_1$ and/or $y = L_2$

$$\lim_{x \to \infty} f(x) = L_1 \qquad \lim_{x \to -\infty} f(x) = L_2$$

Oblique Asymptotes:

1. Apply division of polynomials to the rational expression to break the rational expression into a sum of a polynomial part and a rational part: f(x) = P(x) + R(x).

2. Find the following limits:

$$\lim_{x \to \infty} R(x) \qquad \lim_{x \to -\infty} R(x)$$

3. If both of the limits are 0, we have an oblique asymptote of y = P(x).

Limits at Infinity/Finding Horizontal Asymptotes

In this chapter, we have been examining the first and second derivatives of a function in order to determine various features of the function's graph (relative and absolute extrema, intervals of increasing/decreasing, concavity, etc...). We will now use limits at infinity to examine the end behavior of functions and find any horizontal or slant asymptotes. We will then be reasonably equipped to sketch the graphs for a wide variety of functions.

Limits at Infinity

Examine the graph of the function $f(x) = \dfrac{3x^2}{x^2+1}$ with your graphing calculator.

Notice that as x increases or decreases without bound, the function $f(x)$ approaches ____.

This is denoted by

$\lim_{x \to \infty} f(x) =$ and $\lim_{x \to -\infty} f(x) =$

Notice that we are taking the limit of a function as x approaches infinity or negative infinity. These types of limits are called **limits at infinity**.

Our function approaches the line $y =$ ____ as x increases or decreases without bound. This line is called a horizontal asymptote of the graph of *f*.

Horizontal Asymptotes

The line $y = L$ is a horizontal asymptote of the graph of *f* if

$\lim_{x \to -\infty} f(x) = L$ or $\lim_{x \to \infty} f(x) = L$

Strategy for finding Horizontal Asymptotes

Limits at infinity many of the same properties as the limits that we have discussed before. In addition, the following theorem is useful.

If r is a positive rational number and c is any real number:

1. $\lim_{x\to\infty}\frac{c}{x^r}=0$ and $\lim_{x\to-\infty}\frac{c}{x^r}=0$

In general, if taking the infinite limit yields a constant in the numerator and infinity (or negative infinity) in the denominator, then the limit is 0.

Examples:

Find any horizontal asymptotes of the following functions:

a) $f(x)=5-\frac{2}{x^2}$

b) $f(x)=\frac{2x+5}{3x^2+1}$

c) $f(x)=\frac{2x^2+5}{3x^2+1}$

d) $f(x)=\frac{2x^3+5}{3x^2+1}$

Rational Functions

Many of the functions that you will be examining are rational functions. There are three possible cases, illustrated by the examples b, c, and d on the previous page. Rational functions will have the same limit at infinity and negative infinity.

Type A: If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.

Ex. $\lim_{x \to \pm\infty} \frac{2x+5}{3x^2+1} =$ Horizontal Asymptote:

Type B: If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

Ex. $\lim_{x \to \pm\infty} \frac{2x^2+5}{3x^2+1} =$ Horizontal Asymptote:

Type C: If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function does not exist (it is infinite).

Ex. $\lim_{x \to \pm\infty} \frac{2x^3+5}{3x^2+1} =$ Horizontal Asymptote:

*This last type of rational function actually has a slant asymptote, which we will discuss later.

Example:

Find horizontal asymptotes for each of the following functions:

a) $f(x) = \frac{3-2x}{3x^3-1}$

b) $f(x) = \frac{3-2x}{3x-1}$

c) $f(x) = \frac{3-2x^2}{3x-1}$

Slant Asymptotes

Let us examine a "Type C" case, where the infinite limit of a rational function does not exist (is infinite).

$$f(x) = \frac{2x^2 - 4x}{x+1}$$

This function has no horizontal asymptote, but it does have a *slant* asymptote.
We can find it by using long division to rewrite the improper rational function.

$$x+1\overline{)2x^2 - 4x}$$

$$\frac{2x^2 - 4x}{x+1} =$$

As x approaches infinity or negative infinity, $\frac{6}{x+1}$ becomes very small (and thus insignificant) while $2x-6$ becomes very large (and thus very significant).

Therefore, as x increases or decreases without bound the function $f(x) = \frac{2x^2 - 4x}{x+1}$ looks very much like the function $g(x) = 2x - 6$ and $y = 2x - 6$ is slant asymptote of $f(x)$.

Use your calculator to sketch a graph of $f(x)$ to illustrate this concept.

Limits Approaching Infinity - Horizontal and Slant Asymptotes

1. $\lim_{x\to\infty} \frac{x^a}{x^b}$, $if\ a<b$. Then, limit = 0. Horizontal Asymptote at y = 0.

2. $\lim_{x\to\infty} \frac{Cx^a}{Dx^b}$, $if\ a=b$. Then, limit = $\frac{C}{D}$. Horizontal Asymptote at y = C/D.

3. $\lim_{x\to\infty} \frac{x^a}{x^b}$, $if\ a>b$. Then, limit = ∞. Limit = ∞ indicates that there is no horizontal asymptote.
Slant Asymptote: Find the equation through long division (the remainder is considered to be 0).

Limits Equal to Infinity - Vertical Asymptotes

1. f(a) does not exist but $\lim_{x\to a} f(x)$ = a constant value. "Hole" at x = a (no vertical asymptote).

2. f(a) does not exist and $\lim_{x\to a} f(x)$ = +∞, -∞, or DNE. There is a vertical asymptote at x = a.

For each equation:

a. Determine where the domain does not exist. At these points find the equation of the vertical asymptote or the coordinates of a "hole" in the function.
b. Find the limit approaching infinity. Use this calculation to write the equation of the horizontal asymptote or use long division to find the equation of the slant asymptote.

1. $f(x) = \frac{x^3 - 9x}{3x^2 - 6x - 9}$

2. $f(x) = \frac{x - 4}{-4x - 16}$

3. $f(x) = \frac{1}{x^2 + 1}$

4. $f(x) = \frac{5x^2}{10 - 3x^2}$

5. $f(x) = \frac{x^3 - x^2 - 6x}{-3x^2 - 3x + 18}$

6. $f(x) = \frac{x^3 - 16x}{-4x^2 + 4x + 24}$

7. $f(x) = \frac{1}{3x^2 + 3x - 18}$

8. $f(x) = \frac{3x^2 - 12x}{x^2 - 2x - 3}$

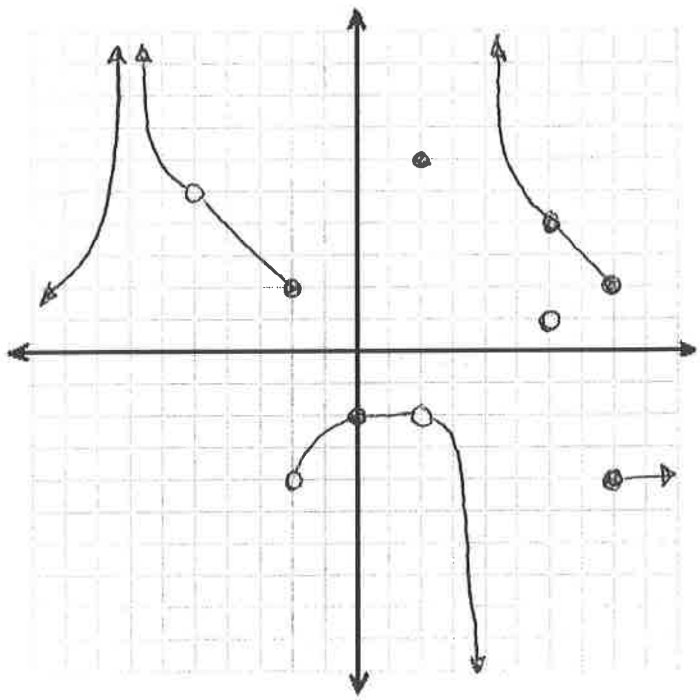
9. $f(x) = \frac{3}{x - 2}$

10. $f(x) = \frac{x^2 + x - 6}{-4x^2 - 16x - 12}$

Continuity Assignment Name: ______________________

1. How is the concept of a limit related to the concept of continuity?

2. List all x values where the graph is discontinuous.
   Explain why discontinuity occurs at each location.



3. Would the graph of $x^2 + y^2 = 7$ be continuous at all points? Why or why not?

4. Give an example of a trig function where discontinuity can be removed.
   Provide a graph of this function.

For each function, list the intervals of continuity.

5. $f(x) = \sqrt[3]{(x + 2)}$

6. $f(x) = \sqrt{(3x)}$

7. $f(x) = \ln(4 - x^2)$

Note all points of discontinuity in each function.
Remove discontinuity when possible by creating a piece-wise function.
If discontinuity cannot be removed, explain why.

8. $f(x) = \frac{x^2 + 2x - 3}{x + 3}$

9. $f(x) = \frac{x^2}{x}$

10. $f(x) = \frac{3}{x^4}$

Using limits to identify asymptotes assignment   Name: ____________________

For each function:

1. Identify any domain restrictions. At these restrictions, find the x value of any holes and /or the equations of any vertical asymptotes.
2. Use the concept of limits at infinity to determine whether there is a horizontal or slant asymptotes. Find the equation of this asymptote.

1. $f(x)=\dfrac{x^3-x^2-6x}{-3x^2-3x+18}$

2. $f(x)=\dfrac{x+4}{-2x-6}$

3. $f(x)=\dfrac{x^3-16x}{-4x^2+4x+24}$

4. $f(x)=-\dfrac{4}{x^2-3x}$