The Antiderivative and Indefinite Integration

Consider the function f(x) = 2x.

If f(x) is the end result of differentiating a function, what was the original function? x^2

We will call this function F(x), the antiderivative of f(x) = 2x.

Definition of Antiderivative

The function F is an antiderivative of the function f on the interval I if F'(x) = f(x) for all x in I.

Consider the function f(x) = cos x.

Find F(x) using your derivative rule page. F(x) = sin x

This antiderivative is not unique, though.
Try to find another function whose derivative would be cos x.

Examples: sin x + 3 sin x − 1

As the derivative of any constant is zero, we need to show that a constant may exist in the antiderivative.

To show that a constant may exist and that the exact constant cannot be found, add "C" to every antiderivative.

Therefore, if f(x) = cos x, then F(x) = sin x + C

Theorem 1 — Functions with Identical Derivatives

Two functions have the same derivative on the interval [a, b] if and only if the functions differ by a constant on the interval [a, b],

$$f'(x) = g'(x) \text{ iff } F(x) = G(x) + C.$$

Definition of indefinite integral

Suppose F is a function such that F'(x) = f(x). The indefinite integral of f(x),

denoted by $\int f(x)\,dx$,

is the most general antiderivative of f(x), which includes an arbitrary constant of integration:

$$\int f(x)\,dx = F(x) + C.$$

The indefinite integral gives a generic description of every possible antiderivative.

For every differentiation fact, there is a corresponding antidifferentiation fact. An antidifferentiation (or integration) formula can be obtained by simply reading a differentiation fact backward. For example,

$$\frac{d}{dx} x = 1 \rightarrow \int 1\,dx = x + C$$

$$\frac{d}{dx} \sin x = \cos x \rightarrow \int \cos x\,dx = \sin x + C$$

You will be a given a rule page for integration similar to the rule page for differentiation.

One of the most frequently used rules for finding derivatives is the power rule.

What are the steps for using the power rule?

1. multiply the exponent to the coefficient : new coefficient
2. reduce the exponent by 1 : new exponent

If integration is the reverse process, the power rule for integration would follow what steps?

1. add 1 to the exponent : new exponent
2. divide the coefficient by the new exponent : new coefficient

There is one value of the exponent for which the power rule will not work. This value is: -1

Use the above process to find the antiderivative of $f(x) = x^4 + 3x^2 + 5$.

$\frac{1}{5}x^5 + x^3 + 5x + C$

Evaluate each of the following indefinite integrals.

(a) $\int 5t^3 - 10t^{-6} + 4\, dt$

$\frac{5}{4}t^4 + 2t^{-5} + 4t + C$

$\frac{5}{4}t^4 + \frac{2}{t^5} + 4t + C$

(b) $\int x^8 + x^{-8}\, dx$

$\frac{1}{9}x^9 - \frac{1}{7}x^{-7} + C$

$\frac{1}{9}x^9 - \frac{1}{7x^7} + C$

(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\, dx$

rewrite: $3x^{3/4} + 7x^{-5} + \frac{1}{6}x^{-1/2}$

$\uparrow$ $7/4$ $\uparrow$ $1/2$

$\frac{12}{7}x^{7/4} - \frac{7}{4}x^{-4} + \frac{1}{3}x^{1/2}$

$\frac{12}{7}\sqrt[4]{x^7} - \frac{7}{4x^4} + \frac{1}{3}\sqrt{x} + C$

(d) $\int dy$

this is $\int 1\, dy$

the deriv. of $x = 1$

$\therefore$

$y + C$

(e) $\int (w + \sqrt[3]{w})(4 - w^2)\, dw$

FOIL and rewrite: $4w + w^3 + 4\sqrt[3]{w} + w^{7/3}$

keep going: $4w + w^3 + 4w^{1/3} + w^{7/3}$

$2w^2 + \frac{1}{4}w^4 + 3w^{4/3} + \frac{3}{10}w^{10/3}$

$2w^2 + \frac{1}{4}w^4 + 3\sqrt[3]{w^4} + \frac{3}{10}\sqrt[3]{w^{10}} + C$

(f) $\int \frac{4x^{10} - 2x^4 + 15x^2}{x^3}\, dx$

rewrite as separate fractions:

$4x^7 - 2x + \frac{15}{x}$

keep going: $4x^7 - 2x + 15 \cdot \frac{1}{x}$

(exponent -1)

$\frac{1}{2}x^8 - x^2 + 15\ln|x| + C$

$\uparrow$

place in abs. value. cannot take ln of neg.

(g) $\int 3e^x + 5\cos x - 10\sec^2 x\, dx$

$3e^x + 5\sin x - 10\tan x + C$

(h) $\int 2\sec w \tan w + \frac{1}{6w}\, dw$

rewrite: $2\sec w \tan w + \frac{1}{6} \cdot \frac{1}{w}$

$2\sec w + \frac{1}{6}\ln|w| + C$

(i) $\int \frac{23}{y^2+1}+6\csc y\cot y+\frac{9}{y}dy$

rewrite:

$23\cdot\frac{1}{y^2+1}+6\csc y\cot y+9\cdot\frac{1}{y}$

$23\tan^{-1}y-6\csc y+9\ln|y|+C$

(j) $\int \frac{3}{\sqrt{1-x^2}}+6\sin x+10\sinh x\,dx$

rewrite:

$3\cdot\frac{1}{\sqrt{1-x^2}}+6\sin x+10\sinh x$

$3\sin^{-1}x-6\sin x+10\cosh x+C$

(k) $\int \frac{7-6\sin^2\theta}{\sin^2\theta}d\theta$

rewrite as 2 fractions: $\frac{7}{\sin^2\theta}-6$

keep going: $7\csc^2\theta-6$

$-7\cot\theta-6x+C$

(l) $\int \sin(x)+10\csc^2(x)dx$

$-\cos x-10\cot x+C$

Given the following information determine the function $f(x)$.

(a) $f'(x)=4x^3-9+2\sin x+7e^x,\ f(0)=15$

1. Find anti-derivative: $x^4-9x-2\cos x+7e^x+C$

2. Use given point to find exact "C": $15=(0)^4-9(0)-2\cos(0)+7e^0+C$

$15=0-0-2+7+C$

$C=10$

$f(x)=x^4-9x-2\cos x+7e^x+10$

(b) $f''(x)=15\sqrt{x}+5x^3+6,\ f(1)=-\frac{5}{4},\ f(4)=404$

$15x^{1/2}+5x^3+6$

1. find $f'(x)$: $10x^{3/2}+\frac{5}{4}x^4+6x+C$

2. find $f''(x)$ - use "D" as 2nd constant: $4x^{5/2}+\frac{1}{4}x^5+3x^2+Cx+D$

3. plug in:

$-\frac{5}{4}=4(1)^{5/2}+\frac{1}{4}(1)^5+3(1)^2+C(1)+D \Rightarrow -\frac{5}{4}=\frac{29}{4}+C+D$

$404=4(4)^{5/2}+\frac{1}{4}(4)^5+3(4)^2+C(4)+D \Rightarrow 404=432+4C+D$

solve system: $C=-\frac{13}{2}$, $d=-2$

$f(x)=4\sqrt{x^5}+\frac{1}{4}x^5+3x^2-\frac{13}{2}x-2$

Integrals

Basic Properties/Formulas/Rules

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant. $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$ where $F(x) = \int f(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $\int_a^b c\,dx = c(b-a)$

If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

If $f(x) \geq g(x)$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Common Integrals

Polynomials

$\int dx = x + c$ $\int k\,dx = k\,x + c$ $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int \frac{1}{x}\,dx = \ln|x| + c$ $\int x^{-1}\,dx = \ln|x| + c$ $\int x^{-n}\,dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$ $\int x^{\frac{p}{q}}\,dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$

Trig Functions

$\int \cos u\,du = \sin u + c$ $\int \sin u\,du = -\cos u + c$ $\int \sec^2 u\,du = \tan u + c$

$\int \sec u \tan u\,du = \sec u + c$ $\int \csc u \cot u\,du = -\csc u + c$ $\int \csc^2 u\,du = -\cot u + c$

$\int \tan u\,du = \ln|\sec u| + c$ $\int \cot u\,du = \ln|\sin u| + c$

$\int \sec u\,du = \ln|\sec u + \tan u| + c$ $\int \sec^3 u\,du = \frac{1}{2}\left(\sec u \tan u + \ln|\sec u + \tan u|\right) + c$

$\int \csc u\,du = \ln|\csc u - \cot u| + c$ $\int \csc^3 u\,du = \frac{1}{2}\left(-\csc u \cot u + \ln|\csc u - \cot u|\right) + c$

Exponential/Logarithm Functions

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$ $\int a^u\,du = \frac{a^u}{\ln a} + c$ $\int \ln u\,du = u\ln(u) - u + c$

$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\sin(bu) - b\cos(bu)\right) + c$ $\int u\mathbf{e}^u\,du = (u-1)\mathbf{e}^u + c$

$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c$ $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + c$

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Examples: ________________ ________________

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Therefore, if $f(x) = \cos x$, then $F(x) =$ ______________

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(c) $\int 3\sqrt[4]{x^3} + \frac{7}{x^5} + \frac{1}{6\sqrt{x}}\,dx$

(d) $\int dy$

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