The Substitution Rule for Indefinite Integration

Pages #1 - #3
Day 1

Show "b" on Page 4
HW: a, c, d on pg. 4
(discuss strategies)

Review HW
15 min to try pg. 5
review pg. 5
Show "a" + "d" on Page 6
HW: b, c, e, f on pg 6
(discuss strat.)
Day 2

Day 3
Review HW
complete page 7

Day 4
graded assignment

After the last section we now know how to do the following integrals.

$$\int \sqrt[4]{x}\, dx \qquad \int \frac{1}{t^3} dt \qquad \int \cos w\, dw \qquad \int e^y\, dy$$

However, we can't do the following integrals.

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx \qquad \int \frac{2t^3+1}{\left(t^4+2t\right)^3} dt$$

$$\int \left(1-\frac{1}{w}\right)\cos\left(w-\ln w\right) dw \qquad \int (8y-1)\, e^{4y^2-y}\, dy$$

All of these look considerably more difficult than the first set. However, they aren't too bad once you see how to do them using the substitution rule for integration. On the first example:

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx$$

1. Similar to the chain rule, we will set the inside equal to "u"

u = $6x^3+5x$

2. Compute the derivative of u

du = $18x^2$ ~~dx~~

3. Now, let's go back to our integral and notice that we can eliminate every x that exists in the integral and write the integral completely in terms of u using both u and du.

$$\int 18x^2 \sqrt[4]{6x^3+5}\, dx \rightarrow \int \sqrt[4]{u}\, du$$

$$4/5 \sqrt[4]{u^5}$$

In the process of doing this we've taken an integral that looked very difficult and with a quick substitution we were able to rewrite the integral into a very simple integral that we can do.

5. Evaluate the integral and "back substitute" and get the integral back into proper form.

$$F(x) = 4/5 \sqrt[4]{(6x^3+5x)^5} + C$$

PAGE #1
Day 1

The Substitution Rule for Integration:

$$\int f(g(x))g'(x)dx = \int f(u)\,du, \qquad \text{where, } u = g(x)$$

Examples

"One Inside Function" → Evaluate each of the following integrals.

(a) $\int \left(1-\frac{1}{w}\right)\cos\underbrace{(w-\ln w)}_{u}\,dw$

$du = 1 - \frac{1}{w}$

$\int \cos u \, du$

$\sin u$

$F(w) = \sin(w - \ln w) + C$

(b) $\int 3(8y-1)\,e^{\underbrace{4y^2-y}_{u}}\,dy$

$du = 8y-1$

$\int 3e^u du$

$3e^u$

$F(y) = 3e^{4y^2-y} + C$

(c) $\int x^2\underbrace{(3-10x^3)}_{u}{}^4\,dx$

$du = -30x^2$

↑

not included - use reciprocal

$\int -\frac{1}{30} u^4 du$

$-\frac{1}{150} u^5$

$F(x) = -\frac{1}{150}(3-10x^3)^5 + C$

(d) $\int \frac{x}{\sqrt{\underbrace{1-4x^2}_{u}}}\,dx$

$du = -8x$

$\int -\frac{1}{8} \cdot \frac{1}{\sqrt{u}}\, du$

$-\frac{1}{4}\sqrt{u}$

$F(x) = -\frac{1}{4}\sqrt{1-4x^2} + C$

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Day 1

"Two Inside Functions" → Evaluate each of the following integrals.

(a) $\int \sin(1-x)\big(\underbrace{2-\cos(1-x)}_{u}\big)^4\,dx$

$du = \sin(1-x)\cdot -1$

$= -\sin(1-x)$

$\int -u^4\,du$

$-\frac{1}{5}u^5$

$F(x) = -\frac{1}{5}\left(2-\cos(1-x)\right)^5 + C$

(b) $\int \cos(3z)\sin^{10}(3z)\,dz$

$= \int \cos(3z)\left[\underbrace{\sin(3z)}_{u}\right]^{10} dz$

$du = 3\cos(3z)$

$\int \frac{1}{3}u^{10}\,du$

$\frac{1}{33}u^{11}$

$F(z) = \frac{1}{33}\left(\sin(3z)\right)^{11}$

$F(z) = \frac{1}{33}\sin^{11}(3z) + C$

(c) $\int \sec^2(4t)\big(\underbrace{3-\tan(4t)}_{u}\big)^3\,dt$

$du = -4\sec^2(4t)$

$\int -\frac{1}{4}u^3\,du$

$-\frac{1}{16}u^4$

$F(t) = -\frac{1}{6}\left(3-\tan(4t)\right)^4 + C$

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Day 1

The most important thing to remember in substitution problems is that after the substitution all the original variables need to disappear from the integral. After the substitution the only variables that should be present in the integral should be the new variable from the substitution (usually u). Note as well that this includes the variables in the differential.

This next set of examples, while not particular difficult, can cause trouble if we aren't paying attention to what we're doing.

Evaluate each of the following integrals.

HW Day 1 **(a)** $\int \frac{3}{\underbrace{5y+4}_{u}}\,dy$

$du = 5$

$\int \frac{3}{5u}\,du$

$\frac{3}{5}\ln|u|$

$\frac{3}{5}\ln|5y+4| + C$

example Day 1 **(b)** $\int \frac{3y}{\underbrace{5y^2+4}_{u}}\,dy$

$du = 10y\,u$

$\int \frac{3}{10y\,u}\,du$

$\frac{3}{10}\ln|u|$

$\frac{3}{10}\ln|5y^2+4| + C$

HW Day 1 **(c)** $\int \frac{3y}{\left(\underbrace{5y^2+4}_{u}\right)^2}\,dy$

$du = 10u$

$\int \frac{3}{10u^2}\,du$

$-\frac{3}{10u}$

$-\frac{3}{10(5y^2+4)^2} + C$

HW Day 1 **(d)** $\int \frac{3}{\underbrace{5y^2+4}_{u}}\,dy$

$du = 5y$

no y

try to make $\frac{1}{x^2+1}$

$\int \frac{3}{4\left(\frac{5}{4}y^2+1\right)} \Rightarrow \frac{3}{4}\int \frac{1}{\left(\frac{\sqrt{5}y}{2}\right)^2+1}$

u

$du = \frac{\sqrt{5}}{2}$

$\int \frac{3}{2\sqrt{5}} \cdot \frac{1}{u^2+1}\,du$

$\frac{3}{2\sqrt{5}}\tan^{-1}u$

$F(y) = \frac{3}{2\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}y}{2}\right) + C$

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Assign Day 1

Review Day 2

In this last set of integrals we had four integrals that were similar to each other in many ways and yet all either yielded different answer using the same substitution or used a completely different substitution than one that was similar to it.

This is a fairly common occurrence and so you will need to be able to deal with these kinds of issues. There are many integrals that on the surface look very similar and yet will use a completely different substitution or will yield a completely different answer when using the same substitution.

Let's take a look at another set of examples to give us more practice in recognizing these kinds of issues.

Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{2t^3+1}{(\underbrace{t^4+2t}_{u})^3}\,dt$

$du = 4t^3+2 = 2(2t^3+1)$

$\displaystyle\int \frac{du}{2u^3} \Rightarrow \int \tfrac{1}{2}u^{-3}\,du$

$-\tfrac{1}{4}u^{-2} \Rightarrow -\dfrac{1}{4u^2}+C$

$-\dfrac{1}{4(t^4+2t)^2}+C$

(b) $\displaystyle\int \frac{2t^3+1}{\underbrace{t^4+2t}_{u}}\,dt$

$du = 4t^3+2 = 2(2t^3+1)$

$\displaystyle\int \frac{du}{2u}$

$= \tfrac{1}{2}\ln|u| + C$

$= \tfrac{1}{2}\ln|t^4+2t| + C$

(c) $\displaystyle\int \frac{x}{\sqrt{\underbrace{1-4x^2}_{u}}}\,dx$

$du = -8x$

$\displaystyle\int \frac{du}{-8\sqrt{u}} \Rightarrow \int -\tfrac{1}{8}u^{-1/2}\,du$

$-\tfrac{1}{4}u^{1/2}+C \Rightarrow -\tfrac{1}{4}\sqrt{u}+C$

$= -\tfrac{1}{4}\sqrt{1-4x^2}+C$

(d) $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

↳ $\frac{d}{dx}$ of $1-4x^2 = -8x$

no x in numerator

use $\sin^{-1}$ instead

try to make $\dfrac{1}{\sqrt{1-x^2}}$

$= \displaystyle\int \frac{1}{\sqrt{1-(\underbrace{2x}_{u})^2}}\,dx$

$du = 2$

$\displaystyle\int \frac{1}{2\sqrt{1-u^2}}\,du \Rightarrow \tfrac{1}{2}\sin^{-1}u + C$

$= \tfrac{1}{2}\sin^{-1}(2x) + C$

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Day 2

A few more examples → Evaluate each of the following integrals.

Example Day 2 (a) $\int e^{2t} + \sec(2t)\tan(2t)\,dt$

$u = 2t$

$du = 2$

$\int \frac{1}{2}(e^u + \sec u \tan u)\,du$

$\frac{1}{2}(e^u + \sec u) + C$

$= \frac{1}{2}(e^{2t} + \sec(2t)) + C$

HW Day 2 (b) $\int \sin(t)(4\cos^3(t) + 6\cos^2(t) - 8)\,dt$

$u = \cos t$

$du = -\sin t$

$\int -(4u^3 + 6u^2 - 8)\,du$

$-(u^4 + 2u^3 - 8u) + C$

$-\cos^4 t - 2\cos^3 t + 8\cos t + C$

HW Day 2 (c) $\int x^2 + e^{1-x}\,dx$

ok needs substitution

$\int x^2 dx + \int e^{1-x}\,dx$

$u = 1 - x \quad du = -1$

$\int x^2 dx + \int -e^u\,du$

$\frac{1}{3}x^3 - e^u + C$

$\frac{1}{3}x^3 - e^{1-x} + C$

Example Day 2 (d) $\int \sin w\sqrt{1 - 2\cos w} + \frac{1}{7w+2}\,dw$

$\int \sin w\sqrt{1 - 2\cos w}\,dw + \int \frac{1}{7w+2}\,dw$

$u = 1 - 2\cos w \qquad u = 7w + 2$

$du = 2\sin w \qquad du = 7$

$\int \frac{1}{2}\sqrt{u}\,du + \int \frac{1}{7u}\,du$

$\frac{1}{3}u^{3/2} = \frac{1}{3}\sqrt{u^3} + \frac{1}{7}\ln|u|$

$\frac{1}{3}\sqrt{(1-2\cos w)^3} + \frac{1}{7}\ln|7w+2| + C$

HW Day 2 (e) $\int \frac{10x+3}{x^2+16}\,dx$

$\underbrace{x^2+16}_{u}$ $du = 2x$ does not match

$\int \frac{10x}{x^2+16}\,dx + \int \frac{3}{x^2+16}\,dx$

$u = x^2 + 16$

$du = 2x$

match $\tan^{-1}$

$\int \frac{3}{16(\frac{1}{16}x^2)+1}$

$\int \frac{5}{u}\,du$

$\int \frac{3}{16(\frac{1}{4}x)^2+1}$

$5\ln|u|$

$u = \frac{1}{4}x \quad du = \frac{1}{4}$

$\int \frac{3}{4}\cdot\frac{1}{u^2+1}\,du$

$\frac{3}{4}\tan^{-1}u$

$5\ln|x^2+16| + \frac{3}{4}\tan^{-1}(\frac{1}{4}x) + C$

HW Day 2 (f) $\int \tan x\,dx$

rewrite as $\int \frac{\sin x}{\cos x}\,dx$

$u = \cos x \quad du = -\sin x$

$\int -\frac{1}{u}\,du$

$-\ln|u|$

$-\ln|\cos x| + C$

or

$\ln|\cos x|^{-1} + C$

$=$

$\ln|\sec x| + C$

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(g) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

$u = x^{1/2}$ $\quad du = \frac{1}{2\sqrt{x}}$

$\int 2\cos u\,du$

$2\sin u$

$2\sin(\sqrt{x}) + C$

* (h) $\displaystyle\int e^{t+e^t}\,dt$

$u = t + e^t$ $\quad du = 1 + e^t$ – does not work

$\int e^t e^{e^t}\,dt$

$u = e^t$ $\quad du = e^t$

$\int e^u\,du$

$e^u = e^{e^t} + C$

HW * (i) $\displaystyle\int \frac{1}{x\ln x}\,dx$

$u = \ln x$ $\quad du = 1/x$

$\int \frac{1}{u}\,du$

$\ln|u|$

$\ln|\ln x| + C$

HW (j) $\displaystyle\int \frac{e^{2t}}{1+e^{2t}}\,dt$

$u = 1 + e^{2t}$ $\quad du = 2e^{2t}$

$\int \frac{1}{2u}\,du$

$\frac{1}{2}\ln|u|$

$\frac{1}{2}\ln|1+e^{2t}| + C$

* (k) $\displaystyle\int \frac{e^{2t}}{1+e^{4t}}\,dt$ $\quad$ 4t does not match 2t

$u = e^{2t}$ $\quad du = 2e^{2t}$

$\int \frac{e^{2t}}{1+(e^{2t})^2}$

$\int \frac{1}{2}\cdot\frac{1}{1+u^2}\,du$

$\frac{1}{2}\tan^{-1} u$

$\frac{1}{2}\tan^{-1}(e^{2t}) + C$

(l) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$u = \sin^{-1}x$ $\quad du = \frac{1}{\sqrt{1-x^2}}$

$\int u\,du$

$\frac{1}{2}u^2$

$\frac{1}{2}(\sin^{-1}x)^2 + C$

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DAY 3

The Substitution Rule for Indefinite Integration

After the last section we now know how to do the following integrals.

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1. Similar to the chain rule, we will set the inside equal to "u"

u = ____________

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(b) $\int 3(8y-1)\,e^{4y^2-y}\,dy$

(c) $\int x^2(3-10x^3)^4\,dx$

(d) $\int \frac{x}{\sqrt{1-4x^2}}dx$

"Two Inside Functions" → Evaluate each of the following integrals.

(a) $\int \sin(1-x)\left(2-\cos(1-x)\right)^4 \, dx$

(b) $\int \cos(3z)\sin^{10}(3z)\, dz$

(c) $\int \sec^2(4t)\left(3-\tan(4t)\right)^3 \, dt$

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(b) $\displaystyle\int \frac{3y}{5y^2+4}\,dy$

(c) $\displaystyle\int \frac{3y}{\left(5y^2+4\right)^2}\,dy$

(d) $\displaystyle\int \frac{3}{5y^2+4}\,dy$

In this last set of integrals we had four integrals that were similar to each other in many ways and yet all either yielded different answer using the same substitution or used a completely different substitution than one that was similar to it.

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(c) $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$ **(d)** $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx$

A few more examples → Evaluate each of the following integrals.

(a) $\int \mathbf{e}^{2t} + \sec(2t)\tan(2t)\,dt$

(b) $\int \sin(t)\left(4\cos^3(t) + 6\cos^2(t) - 8\right)dt$

(c) $\int x^2 + \mathbf{e}^{1-x}\,dx$

(d) $\int \sin w\sqrt{1 - 2\cos w} + \frac{1}{7w+2}\,dw$

(e) $\int \frac{10x+3}{x^2+16}\,dx$

(f) $\int \tan x\,dx$

(g) $\displaystyle\int \frac{\cos\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

(h) $\displaystyle\int \mathbf{e}^{t+\mathbf{e}^t}\,dt$

(i) $\displaystyle\int \frac{1}{x\ln x}\,dx$

(j) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{2t}}\,dt$

(k) $\displaystyle\int \frac{\mathbf{e}^{2t}}{1+\mathbf{e}^{4t}}\,dt$

(l) $\displaystyle\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$