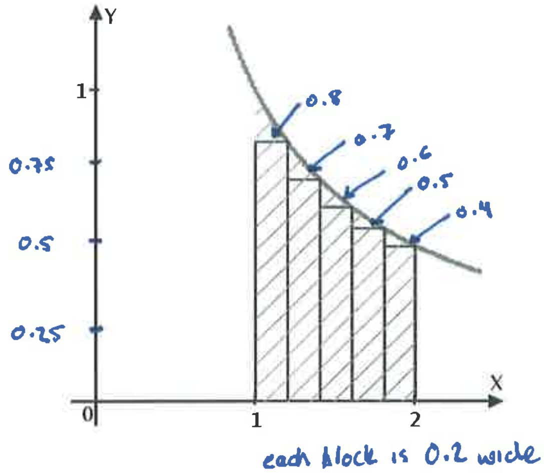
Math 151: Connecting Antiderivatives and Area (5.2-5.3 Worksheet)

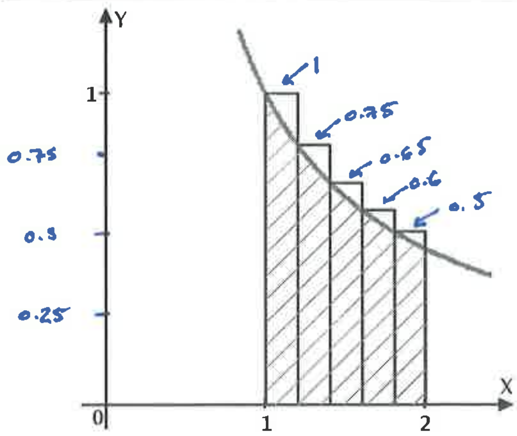
1. Use the five rectangles in the figures below to find **a lower approximation** of the area of the region lying between the graph of $f(x) = \frac{1}{x}$ and the x-axis between $x = 1$ and $x = 2$.



0.8 · 0.2 = 0.16
0.7 · 0.2 = 0.14
0.6 · 0.2 = 0.12
0.5 · 0.2 = 0.10
0.4 · 0.2 = 0.08

0.6

2. Use another five rectangles (but with different heights) to find **an upper approximation** of the area of the region lying between the graph of $f(x) = \frac{1}{x}$ and the x-axis between $x = 1$ and $x = 2$.



1 · 0.2 = 0.2
0.75 · 0.2 = 0.15
0.65 · 0.2 = 0.13
0.6 · 0.2 = 0.12
0.5 · 0.2 = 0.10

0.7

Imagine what would happen if we increased the number of rectangles in the previous situation…

- As the ***number of rectangles*** we use to estimate the area ***INCREASES***, the *width* of each individual rectangle will ___decrease___.
- As the ***number of rectangles*** we use to estimate the area ***INCREASES***, the *accuracy* of the overall area estimate will ___increase___.
- If we let the ***number of rectangles*** we use to estimate the area ***approach infinity***, we have a calculus problem involving ___the limit___ (a concept introduced way back in Chapter 2).

So, what's the connection between areas, functions, and all the concepts we've been studying so far?

Mathematicians discovered an amazing relationship between **areas under curves** (like above) and **antiderivatives**. In plain English – so long as a function is continuous (i.e., no gaps, missing values, or vertical asymptotes) over a specific interval, the antiderivative of the function is the *key* to finding the area under the curve.

continuous functions can be integrated

THEOREM 5.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.

area = the definite integral

THEOREM 5.5 The Definite Integral as the Area of a Region

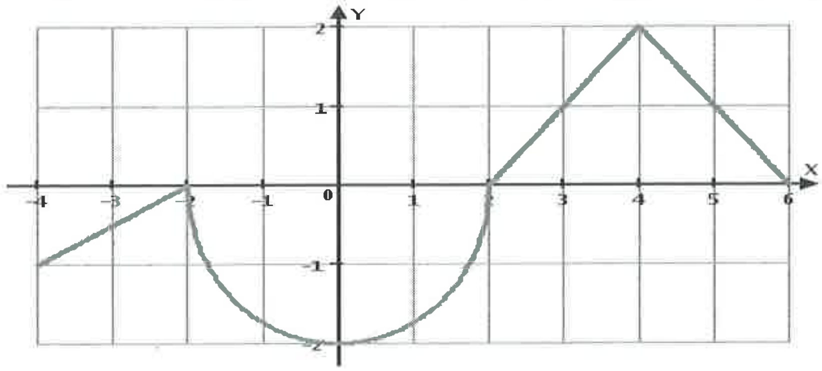
If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 5.22.)

We can generalize this a bit further by saying that "**negative area**" is the result of having a function go ***below* the *x*-axis**. It doesn't really make much sense physically – i.e., how can you have "negative area" in real life? – but this allows our equations to have both size and direction, and it keeps things consistent. much like negative velocity

3. The graph of *f* consists of line segments and a semicircle, as shown below. Evaluate each definite integral by using geometric formulas. *Note: there are some places where f(x) is negative.*



a) $\int_0^2 f(x)dx$

From 0 to 2 is 1/4 circle under x-axis

$A = -\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi \cdot 2^2 = -\pi$

b) $\int_2^6 f(x)dx$

from 2 to 6 is a Δ (above x-axis)

$A = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 4 \cdot 2 = 4$

c) $\int_{-4}^2 f(x)dx$ -4 to 2 Δ + -2 to 2 1/2 circle both are under

$A = -\frac{1}{2} \cdot 2 \cdot 1 + -\frac{1}{2} \cdot \pi \cdot 2^2 = -(1 + 2\pi)$

or $-1 - 2\pi$

d) $\int_{-4}^6 f(x)dx$

-4 to 2 | -2 to 2 | 2 to 6

A = -1 | A = -2π | A = 4

$A = 3 - 2\pi$

e) $\int_{-4}^6 |f(x)|dx$

↑ all positive

-4 to 2 | -2 to 2 | 2 to 6

A = 1 | A = 2π | A = 4

$A = 5 + 2\pi$

Theorem 5.5 (above) is way to connect the integral (antiderivative) to the area between a function's curve and the *x*-axis (between two *x*-values, *a* and *b*). *Be sure to notice* that there are now **TWO** types of integrals – the kind we started with that did not specify and interval, and this new kind that does specify an interval. The first is called an "**indefinite integral**" and the newer, second kind is called a "**definite integral**."

Just like we found with limits, derivatives, and indefinite integrals, there are some **special cases** and "grammar rules" with definite integrals that we have to keep in mind. You'll find them on the next page, along with some examples for you to work out.

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then we define $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

← no width of rectangle...

4. Given $\int_0^5 f(x)dx = 10$, evaluate:

a) $\int_5^0 f(x)dx$

flipped numbers - make negative

$= -\int_0^5 f(x)\,dx = -10$

b) $\int_5^5 f(x)dx$

same start + end pt

$= 0$

THEOREM 5.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

5. Given $\int_0^5 f(x)dx = 10$ and $\int_5^7 f(x)dx = 3$, evaluate $\int_0^7 f(x)dx$.

0 to 5 = 10 5 to 7 = 3

$10 + 3 = 13$

THEOREM 5.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions of kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

6. Evaluate the integrals using the rules and the following values: $\int_2^4 x^3dx = 60$ $\int_2^4 xdx = 6$ $\int_2^4 dx = 2$

a) $\int_2^4 4xdx$

$4\int_2^4 x\,dx$

$4 \cdot 6$

24

b) $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right)dx$

$\frac{1}{2}\int_2^4 x^3dx - 3\int_2^4 xdx + 2\int_2^4 dx$

$\frac{1}{2} \cdot 60 - 3 \cdot 6 + 2 \cdot 2$

$30 - 18 + 4$

16

Unfortunately, we don't always have nice, neat geometric formulae to use or pre-solved definite integrals at hand. Even so, we ***do*** know how to integrate and find antiderivatives. Therefore, all we need is a grammar (or, in our case, arithmetic) for evaluating an integral over a specific interval of values.

The following theorem gives us the grammar to do just that:

THEOREM 5.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

As the name would indicate, this "***Fundamental Theorem of Calculus***" is pretty important – in fact, along with the **definition of derivative**, this is probably the **most important** content for any beginning calculus student to learn and understand!

So, we come around full-circle, back to where we began: trying to calculate area...

Recall the first two problems on page 1 of this worksheet:

- We know we can use collections of rectangles to calculate upper and lower estimates for the area under a curve and above the *x*-axis over a specific interval (see problems #1 and #2).
- We know that if a function is continuous over that specific interval, (according to Theorem 5.4) we can integrate it.
- Also, we know from Section 5.1 how to find the antiderivative of $f(x)$.
- Therefore, we can use Theorem 5.5 to find the *exact* value for the area under the curve (which is connected to the antiderivative via the Fundamental Theorem of Calculus).

This is one of those important moments where we are able to synthesize nearly an entire semester of concepts into one important process. It definitely will take some time and practice in order to get comfortable and fluent with all of this.

The next page walks you through this process using the function, $f(x) = \frac{1}{x}$, which we already know something about from page 1 of this worksheet...

7. Find the area under the curve, $f(x)=\frac{1}{x}$, above the *x*-axis, and between *x* = 1 and *x* =2. That is, in full "definite integral" form, we need to evaluate $\int_1^2\left(\frac{1}{x}\right)dx$.

a. **First Step**: Integrate to find the antiderivative function: $\int\left(\frac{1}{x}\right)dx = F(x)$

formula page: $\int \frac{1}{x}dx = \ln x$

$F(x) = \ln x$

since definite $F(x)\Big|_1^2 \ \ln x$

b. **Second Step**: Evaluate the antiderivative, $F(x)$, at *x* = 1 and *x* =2. That is, find $F(2)$ and $F(1)$.

$F(x)\Big|_1^2 \ \ln x = \ln 2 - \ln 1$

$F(2) = 0.6931$ $\quad F(1) = 0$

c. **Third Step**: Evaluate $F(2)-F(1)$. This will be just a number, since the constant terms cancel out. By virtue of the Fundamental Theorem of Calculus, this is the *exact value* for the area between the curve and the *x*-axis, from 1 to 2.

$F(x)\Big|_1^2 \ \ln x = 0.6931 - 0$

$= \underline{0.6931}$

d. **Fourth Step**: Verify that this answer is between the upper and lower estimates that we found.

✓

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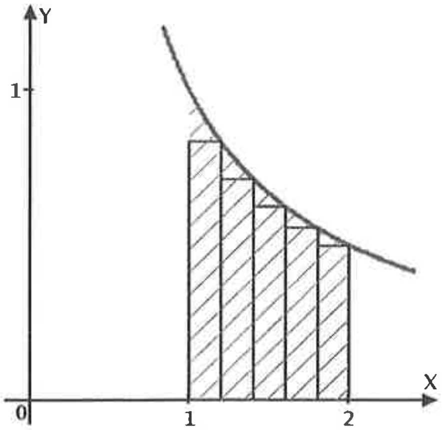

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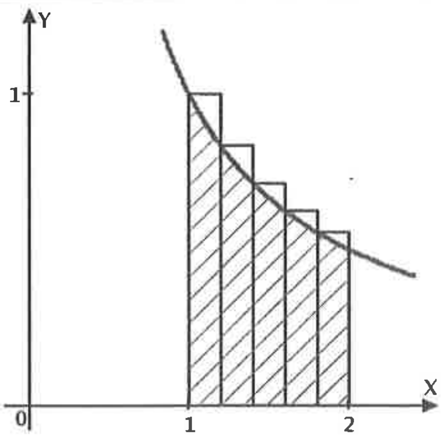

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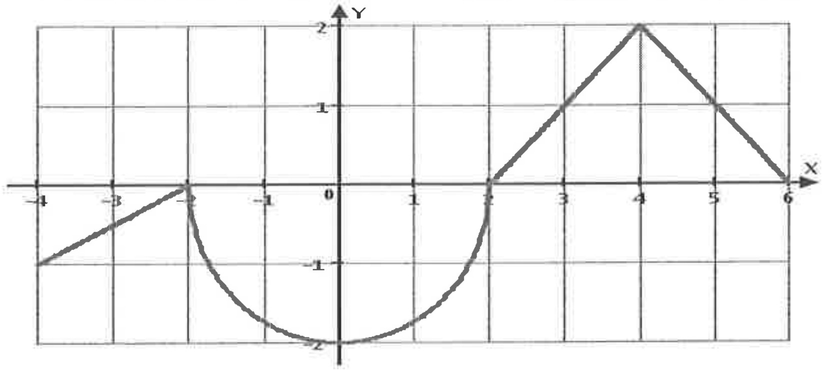

a) $\int_{0}^{2} f(x)dx$

b) $\int_{2}^{6} f(x)dx$

c) $\int_{-4}^{2} f(x)dx$

d) $\int_{-4}^{6} f(x)dx$

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