

Answers to Optimization Worksheet

1. a. The product of 2 positive numbers is being maximized.

$$P = xy$$

b. The constraint is:

The sum of the first and twice the second numbers is 100.

$$x + 2y = 100$$

c. $x = 100 - 2y$

d. $P = xy$

$$P = (100 - 2y)y$$

$$P = 100y - 2y^2$$

e. $P' = 100 - 4y = 0$

| | |
|----------|----------------|
| $y = 25$ | $x = 100 - 2y$ |
| | $x = 50$ |

f. The two numbers are 25 and 50.

2. Minimize: the sum of the first plus three times the second.

$$S = x + 3y$$

Constraint: the product of the two numbers is 192.

$$xy = 192$$

$$y = \frac{192}{x}$$

$$S = x + 3y$$

$$S = x + 3\left(\frac{192}{x}\right)$$

$$S = x + 576x^{-1}$$

$$S' = 1 - \frac{576}{x^2} = 0$$

$$\frac{x^2 - 576}{x^2} = 0$$

$$x^2 - 576 = 0$$

$$x = 24$$

$$y = \frac{192}{24}$$

$$y = 8$$

$$\frac{+576}{x^2} = +1$$

$$x^2 = 576$$

$$x = \pm 24$$

positive makes sense,
negative doesn't

The two numbers
are 24 and 8.

3. a. The area of the rectangular enclosure is being maximized.

$$A = LW$$

b. There is 120m fencing for three walls.



c. $x + 2y = 120$

$$x = 120 - 2y$$

d. $A = LW$

$$A = (x)(y)$$

$$A = (120 - 2y)(y)$$

$$A = 120y - 2y^2$$

e. $A' = 120 - 4y = 0$

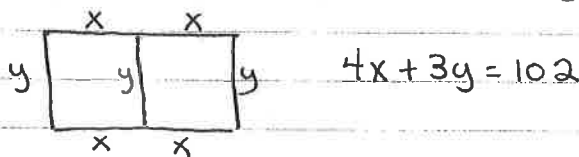
$$y = 30 \text{ m} \quad x = 120 - 2(30) = 60 \text{ m}$$

f. Since $x = 60 \text{ m}$ and $y = 30 \text{ m}$, the maximum area of the enclosure is 180 m^2 .

4. Maximize: area of the enclosure

$$A = LW$$

Constraints: There is 102m fencing for the enclosure.



Because $4x + 3y = 102$
 $y = 34 - \frac{4}{3}x$

$$A = LW$$

$$A = (2x)(y)$$

$$A = (2x)(34 - \frac{4}{3}x)$$

$$A = 68x - \frac{8}{3}x^2$$

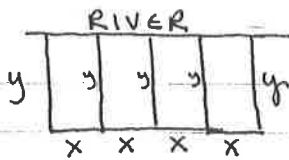
$$A' = 68 - \frac{16}{3}x = 0$$

$$x = \frac{51}{4} \approx 12.75 \text{ m}$$

$$y = 17$$

The maximum area is $2(12.75)(17) = 433.5 \text{ m}^2$

5. Maximize: Area of 4 pens
 Constraint: There is 150 feet of fencing.



$$\begin{aligned} \text{Area} &= 4xy \\ \text{Constraint} &= 4x + 5y = 150 \\ & y = 30 - \frac{4}{5}x \end{aligned}$$

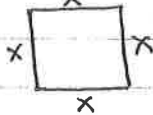
$$\begin{aligned} \text{Area} &= 4xy \\ A &= 4x(30 - \frac{4}{5}x) \\ A &= 120x - \frac{16}{5}x^2 \\ A' &= 120 - \frac{32}{5}x = 0 \end{aligned}$$

$$x = 18.75 \text{ ft.} \quad y = 30 - \frac{4}{5}(18.75) = 15 \text{ ft.}$$

If $x = 18.75 \text{ ft.}$ and $y = 15 \text{ ft.}$, the maximum area of all four pens is 1125 ft^2 .

6. Minimize: Area of the enclosure(s) ^{square} AND Maximize the area.
 Constraint: There is 200 m of fencing to make either one enclosure or two separate ^{square} enclosures.

One enclosure



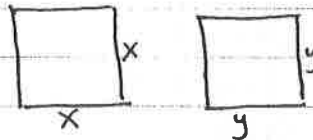
$$\begin{aligned} 4x &= 200 \\ x &= 50 \text{ m} \end{aligned}$$

Therefore

$$A = 50^2 = 2500 \text{ m}^2$$

Max. Area = 2500 m^2
 using one enclosure.
 Min. Area = 1250 m^2
 using two enclosures.

Two separate square enclosures:



$$\begin{aligned} \text{Total Area} &= x^2 + y^2 \\ \text{Constraint: } & 4x + 4y = 200 \end{aligned}$$

$$\begin{aligned} 4x + 4y &= 200 \\ y &= 50 - x \end{aligned}$$

$$A = x^2 + y^2$$

$$A = x^2 + (50 - x)^2$$

$$A = x^2 + 2500 - 100x + x^2$$

$$A = 2x^2 - 100x + 2500$$

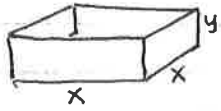
$$A' = 4x - 100 = 0$$

$$x = 25 \text{ m}, y = 50 - 25 = 25 \text{ m}$$

$$\text{Total Area} = 25^2 + 25^2 = 1250 \text{ m}^2$$

7. Maximize: Volume of open box with square base.

Constraint: Surface area = 108 in^2 .



$$\text{Max: } V = x^2 y$$

$$\text{Constraint: } x^2 + 4xy = 108$$

$$y = \frac{27}{x} - \frac{1}{4}x$$

$$V = x^2 y$$

$$V = x^2 \left(\frac{27}{x} - \frac{1}{4}x \right)$$

$$V = 27x - \frac{1}{4}x^3$$

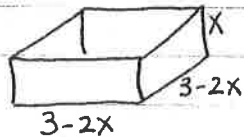
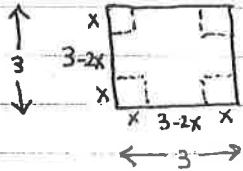
$$V' = 27 - \frac{3}{4}x^2 = 0$$

$$x = 6'' \quad y = \frac{27}{x} - \frac{1}{4}x = 3''$$

The dimensions of the open box are $6'' \times 6'' \times 3''$.

8. Maximize: Volume of open box

Constraint: Must use square piece of cardboard: $3' \times 3'$.



$$V = LWH$$

$$V = (3-2x)(3-2x)(x)$$

$$V = (9 - 12x + 4x^2)(x)$$

$$V = 4x^3 - 12x^2 + 9x$$

$$V' = 12x^2 - 24x + 9 = 0$$

$$= 3(4x^2 - 8x + 3) = 0$$

$$= 3(2x-3)(2x-1) = 0$$

$$x = \frac{3}{2} \text{ or } \frac{1}{2}$$

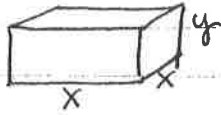
x cannot be $\frac{3}{2}$, because of the constraint: Side of box = $3-2x$

$3 - 2\left(\frac{3}{2}\right) = 0!$ There would be no box

if $x = \frac{3}{2}!$

So, $x = \frac{1}{2}$ and the maximum volume is 2 ft^3 .

9. Maximize: Volume of closed box with square base
 Constraint: Surface Area of box is 100 in^2 .



$$V = LWH$$

$$V = x^2y$$

$$\text{Surface Area} = 2x^2 + 4xy = 100$$

$$y = \frac{25}{x} - \frac{1}{2}x$$

$$V = x^2y$$

$$V = x^2\left(\frac{25}{x} - \frac{1}{2}x\right)$$

$$V = 25x - \frac{1}{2}x^3$$

$$V' = 25 - \frac{3}{2}x^2 = 0$$

$$x = \sqrt{\frac{50}{3}} \approx 4.08 \text{ in.}$$

$$y = \frac{25}{x} - \frac{1}{2}x$$

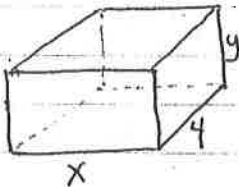
$$y \approx 4.08 \text{ in.}$$

The maximum volume of the box is about 68.04 in^3

10. Minimize: Cost of open rectangular tank.
 Constraint: Base cost = $\$10/\text{m}^2$ and sides cost = $\$5/\text{m}^2$

Width = 4 m

Volume = 36 m^3



$$V = 4xy = 36$$

$$y = \frac{9}{x}$$

base sides sides

$$SA = 4x + (4y)(2) + (xy)(2)$$

$$SA = 4x + 8y + 2xy$$

$$C = (4x)(\$10) + (8y)(\$5) + (2xy)(\$5)$$

$$C = (4x)(10) + (8)\left(\frac{9}{x}\right)(5) + (2x\left(\frac{9}{x}\right))(5)$$

$$C = 40x + 360x^{-1} + 90$$

$$C' = 40 - 360x^{-2}$$

$$C' = 40 - \frac{360}{x^2} = 0$$

$$\frac{360}{x^2} = 40$$

$$x^2 = 9$$

$$x = \pm 3$$

$$40x^2 = 360$$

$$C' = \frac{40x^2 - 360}{x^2} = 0$$

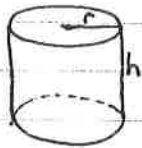
$$40x^2 - 360 = 0$$

$$x = 3$$

$$y = \frac{9}{x} = 3$$

Dimensions are $3 \text{ m} \times 3 \text{ m} \times 4$
 Cost is $\$330$.

11. Minimize: Surface area of cylindrical container.
 Constraint: Volume = 355 ml Note: $1\text{cm}^3 = 1\text{ml}$



Minimize $SA = 2\pi r^2 + 2\pi r h$ (top & bottom, sides)
 Constraint $V = 355 = \pi r^2 h$
 $h = \frac{355}{\pi r^2}$

$SA = 2\pi r^2 + 2\pi r h$
 $SA = 2\pi r^2 + 2\pi r \left(\frac{355}{\pi r^2}\right)$

$SA = 2\pi r^2 + \frac{710}{r}$

$SA = 2\pi r^2 + 710r^{-1}$
 $SA' = 4\pi r - 710r^{-2}$
 $SA' = 4\pi r - \frac{710}{r^2} = 0$

$SA' = 4\pi r = \frac{710}{r^2}$

$r = \sqrt[3]{\frac{710}{4\pi}} \approx 3.84\text{cm}$ $h = \frac{355}{\pi r^2} \approx 7.66\text{cm}$

The dimensions are radius = 3.84cm and height = 7.66cm.

12. Minimize Surface Area of can: $SA = 2\pi r^2 + 2\pi r h$
 Constraint: Top & bottom cost \$2/in²; sides cost \$6/in².
 $V = 300\text{in}^3$



$V = \pi r^2 h = 300$
 $h = \frac{300}{\pi r^2}$

Cost = $2\pi r^2(\$2) + 2\pi r h(\$6)$
 $C = 4\pi r^2 + 12\pi r \left(\frac{300}{\pi r^2}\right)$

$C = 4\pi r^2 + 3600r^{-1}$
 $C' = 8\pi r - \frac{3600}{r^2} = 0$

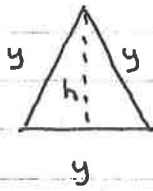
$r = \sqrt[3]{\frac{3600}{8\pi}} \approx 5.23\text{in}$

$h \approx 3.49\text{in}$

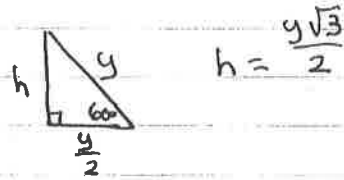
Dimensions are radius = 5.23 in.
 height = 3.49 in.
 Cost \approx \$1032

13. Maximize Area of triangle and square : $A = \frac{1}{2}bh + x^2$

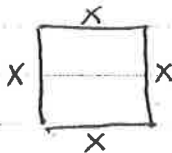
Constraint: Combined perimeter is 10 : $P = 3y + 4x = 10$



← Equilateral triangle!



$$h = \frac{y\sqrt{3}}{2}$$



$$P = 3y + 4x = 10$$

$$x = \frac{10 - 3y}{4}$$

$$\begin{aligned} \text{Area} &= \frac{1}{2}bh + x^2 \\ &= \frac{1}{2}y\left(\frac{y\sqrt{3}}{2}\right) + \left(\frac{10 - 3y}{4}\right)^2 \end{aligned}$$

$$A = \frac{y^2\sqrt{3}}{4} + \frac{100 - 60y + 9y^2}{16}$$

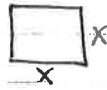
$$A = \frac{y^2\sqrt{3}}{4} + \frac{25}{4} - \frac{15}{4}y + \frac{9}{16}y^2$$

$$A' = \frac{\sqrt{3}}{2}y - \frac{15}{4} + \frac{9}{8}y = 0$$

$$y \approx 1.883 \quad x \approx 1.088$$

The triangle has sides = 1.883.
The square has sides = 1.088.

14. Maximize Area of circle and square: $A = \pi r^2 + x^2$



Constraint: Perimeter of both = 16.

$$P = 2\pi r + 4x = 16$$

$$r = \frac{8-2x}{\pi}$$

$$A = \pi r^2 + x^2$$

$$A = \pi \left(\frac{8-2x}{\pi} \right)^2 + x^2$$

$$A = x^2 + \frac{64}{\pi} - \frac{32x}{\pi} + \frac{4x^2}{\pi}$$

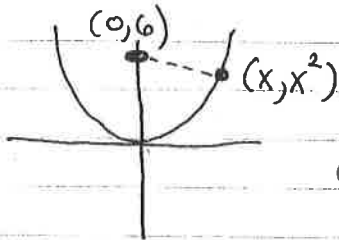
$$A' = 2x - \frac{32}{\pi} + \frac{8}{\pi}x = 0$$

$$x \approx 2.24 \quad r \approx 1.12$$

The sides of the square are 2.24.
The radius of the circle is 1.12.

15. Minimize distance from $(0,6)$ to $y=x^2$.

Constraint: the point $(0,6)$ and $y=x^2$.



$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Our two points are $(0,6)$ and (x,x^2)

$$D = \sqrt{(x-0)^2 + (x^2-6)^2}$$

$$D = (x^2 + x^4 - 12x^2 + 36)^{1/2}$$

$$D = (x^4 - 11x^2 + 36)^{1/2}$$

$$D' = \frac{1}{2}(x^4 - 11x^2 + 36)^{-1/2}(4x^3 - 22x)$$

$$D' = \frac{4x^3 - 22x}{2\sqrt{x^4 - 11x^2 + 36}} = 0 \rightarrow 4x^3 - 22x = 0 \quad x=0, \sqrt{\frac{11}{2}}$$

If $x=0, y=0$ and $D=6$

If $x=\sqrt{\frac{11}{2}}, y=\frac{11}{2}$ and $D=5.75$ minimum distance

1. Find two positive numbers such that the sum of the first and twice the second is 100 and their product is a maximum.

a. What is being maximized or minimized?

words: _____

equation: _____

b. What are the constraints? Do you need to draw a picture?

words: _____

equation: _____

c. Solve the constraint equation for one variable:

d. Use the constraint equation to rewrite the max/min function in terms of one variable and simplify it.

e. Find the critical points. Determine the absolute max or min.

f. Read the problem again, have you answered it? Does your answer make sense in the problem? Write a sentence to answer the question.

2. Find two positive numbers such that their product is 192 and the sum of the first plus three times the second is a minimum.

3. A gardener wants to make a rectangular enclosure using a wall as one side and 120 m of fencing for the other three sides. Find the maximum area of the enclosure.

a. What is being maximized or minimized?

words: _____

equation: _____

b. What are the constraints? Do you need to draw a picture?

words: _____

equation: _____

c. Solve the constraint equation for one variable:

d. Use the constraint equation to rewrite the max/min function in terms of one variable and simplify it.

e. Find the critical points. Determine the absolute max or min.

f. Read the problem again, have you answered it? Does your answer make sense in the problem? Write a sentence to answer the question.

4. Suppose you had 102 m of fencing to make two side-by-side enclosures as shown. What is the maximum area that you could enclose?



5. Four pens will be built side by side along a river by using 150 feet of fencing. What dimensions will maximize the area of the pens?
6. Suppose you had to use exactly 200 m of fencing to make either one square enclosure or two separate square enclosures of any size you wished. What plan would give you the least area? What plan would give you the greatest area?
7. A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?
8. A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
9. A box manufacturer desires to create a box with a surface area of 100 in^2 . What is the maximum volume that can be formed by bending this material into a closed box with a square base, square top, and rectangular sides?
10. A tank with a rectangular base and rectangular sides is open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $\$10/\text{m}^2$ for the base and $\$5/\text{m}^2$ for the sides, what is the cost of the least expensive tank, and what are its dimensions?
11. A regular soda pop can has a diameter of about 6.8 cm with a height of 12.5 cm. It holds 355 ml of soda pop. Find the dimensions of a can that has the same volume but uses the least amount of material to construct. Note: $1 \text{ cm}^3 = 1 \text{ ml}$
12. A cylinder has a volume of 300 in^3 . The top and bottom parts of the cylinder cost $\$2$ per in^2 and the sides of the cylinder cost $\$6/\text{in}^2$. What are the dimensions of the most economical cylinder and how much will it cost?
13. The combined perimeter of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a maximum total area.
14. The combined perimeter of a circle and a square is 16. Find the dimensions of the circle and square that produce a minimum total area.
15. Find the point on the graph of $y = x^2$ that is the smallest distance from the point (0, 6).