Calculus Name Key

Review - Unit #3

Date_______________

Solve each related rate problem.

1) A spherical snowball melts at a rate of $\frac{256\pi}{3}$ in³/sec. At what rate is the radius of the snowball changing when the radius is 3 in?

$\frac{dV}{dt} = -\frac{256\pi}{3}\ \frac{in^3}{sec}$ $\quad \frac{dr}{dt} = ?$ $\quad r=3$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$-\frac{256\pi}{3} = 4\pi(3)^2 \frac{dr}{dt}$

$-\frac{256\pi}{3} = 36\pi \frac{dr}{dt}$ $\quad \frac{dr}{dt} = -\frac{64}{27}\ \frac{in}{sec}$

$\boxed{\text{The radius is decreasing at a rate of } \frac{64}{27} \text{ in/sec.}}$

2) A conical paper cup is 30 cm tall with a radius of 10 cm. The bottom of the cup is punctured so that the water leaks out at a rate of $\frac{16\pi}{3}$ cm³/sec. At what rate is the water level changing when the water level is 2 cm?

$h=30 \quad r=10$ $\quad \frac{dV}{dt} = -\frac{16\pi}{3}\ \frac{cm^3}{sec}$ $\quad \frac{dh}{dt} = ?$ $\quad h=2$

$\frac{30}{h} = \frac{10}{r}$

$30r = 10h$

$r = \frac{1}{3}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V = \frac{1}{3}\pi \frac{1}{9}h^2 \cdot h$

$V = \frac{1}{27}\pi h^3$

$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

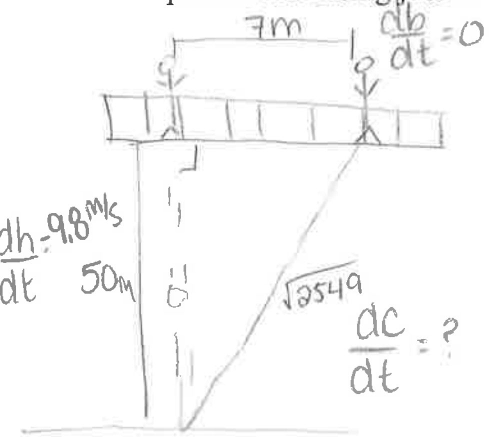
$-\frac{16\pi}{3} = \frac{1}{9}\pi(2)^2 \frac{dh}{dt}$

$-\frac{16\pi}{3} = \frac{4}{9}\pi \frac{dh}{dt}$

$\frac{dh}{dt} = -12$ cm/sec

$\boxed{\text{The water level is decreasing at 12 cm/sec.}}$

3) A rock is dropped straight off a bridge that is 50 meters above the ground. Another person is 7 meters away on the same bridge. At what rate is the distance betwee nthe rock and the second person increasing just as the rock hits the ground?



$50^2 + 7^2 = c^2$

$2500 + 49 = c^2$

$2549 = c^2$

$c = \sqrt{2549}$

$h^2 + b^2 = c^2$

$2h\frac{dh}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

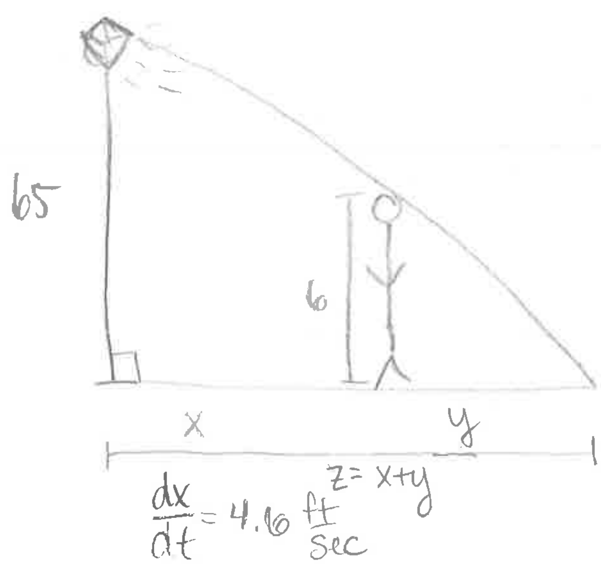
$2(50)(9.8) + 2(7)(0) = 2(\sqrt{2549})\frac{dc}{dt}$

$980 = 2\sqrt{2549}\frac{dc}{dt}$

$\frac{dc}{dt} = \frac{490}{\sqrt{2549}} \approx 9.71$ m/sec

The distance between the rock and the second person is increasing at a rate of approximately 9.71 m/sec.

4) A light shines from the top of a pole 65 ft high. A 6 ft person walks away from the light pole at a rate of 4.6 ft/sec. How fast is the distance from the light post to the tip of the shadow increasing?



$\frac{6}{y} = \frac{65}{x+y}$

$6x + 6y = 65y$

$59y = 6x$

$y = \frac{6x}{59}$

$z = x + y$

$z = x + \frac{6x}{59}$

$z = \frac{65x}{59}$

$\frac{dz}{dt} = \frac{65}{59}\frac{dx}{dt}$

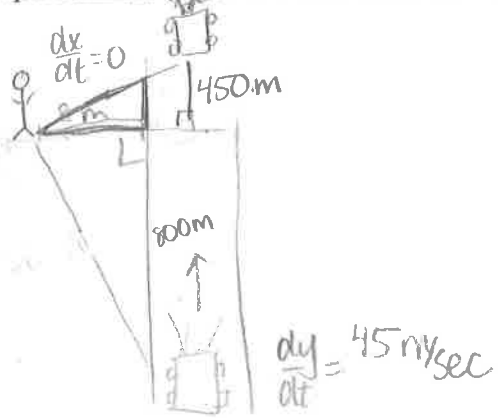
$\frac{dz}{dt} = \frac{65}{59}(4.6)$

$\frac{dz}{dt} \approx 5.067797$

The distance from the light post to the tip of the shadow is increasing at approximately 5.07 ft/sec.

5) A person is 8 meters away from a road and there is a car that is initially 800 meters away approaching the person at a speed of 45 m/sec. At what rate is the distance between the person and the car changing ~~(a) 5 seconds after the start, (b) when the car is directly in front of the person and (c)~~ 10 seconds after the car has passed the person?

$\frac{dz}{dt} = ?$



$8^2 + 450^2 = c^2$

$64 + 202500 = c^2$

$202564 = c^2$

$c = 450.07$

$x^2 + y^2 = z^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$2(8)(0) + 2(450)(45) = 2(450.07)\frac{dz}{dt}$

$40500 = 900.14\frac{dz}{dt}$

$\frac{dz}{dt} = 44.993$

The distance between the person and the car is increasing at a rate of 45 m/sec.

For each problem, find all points of absolute minima and maxima on the given interval.

6) $y = +\frac{-9x}{x^2 + 9}$; $[2, 5]$

$y' = \frac{-9(x^2+9) + 9x(2x)}{(x^2+9)^2} = \frac{-9x^2 - 81 + 18x^2}{(x^2+9)^2}$

$9x^2 - 81 = 0$

$9x^2 = 81$

$x^2 = 9$

$x = \pm 3$

$x^2 + 9 = 0$

$x^2 = -9$

not possible

$y|_2 = \frac{-18}{13}$

$y|_3 = \frac{-3}{2}$

$y|_5 = \frac{-45}{34}$

abs. max: $(5, \frac{-45}{34})$

abs. min: $(3, -3/2)$

Find the x-coordinates of critical points, then find the open intervals where the function is increasing and decreasing.

7) $h(s) = -s^4 + 4s^2 - 4$

$h'(s) = -4s^3 + 8s = 0$

$s(-4s^2 + 8) = 0$

$-4s^2 = -8$

$s^2 = 2$

$s = 0 \quad s = \sqrt{2} \quad s = -\sqrt{2}$

Critical pts

+ | − | + | −

$-\sqrt{2}$ 0 $\sqrt{2}$

Increasing $(-\infty, -\sqrt{2}), (0, \sqrt{2})$

Decreasing $(-\sqrt{2}, 0), (\sqrt{2}, \infty)$

8) $f(w) = -2\sin(2w)$; $[-\pi, \pi]$

$f'(w) = -4\cos(2w) = 0$

$\cos(2w) = 0$

$2w = \frac{\pi}{2} + 2\pi n$

$2w = \frac{3\pi}{2} + 2\pi n$

$w = \frac{\pi}{4} + \pi n$

$w = \frac{3\pi}{4} + \pi n$

Critical pts: $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

$-\pi \quad - \quad -\frac{3\pi}{4} \quad + \quad -\frac{\pi}{4} \quad - \quad \frac{\pi}{4} \quad + \quad \frac{3\pi}{4} \quad - \quad \pi$

$\boxed{\text{Increasing } \left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \quad \text{Decreasing } \left(-\pi, -\frac{3\pi}{4}\right), \left(-\frac{\pi}{4}, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \pi\right)}$

9) $g(w) = \frac{1}{w^2 - 16}$ $\quad (w^2-16)^{-1}$

$g'(w) = -2w(w^2-16) = 0$

$-2w = 0 \quad w^2 - 16 = 0$

$w = 0 \quad w^2 = 16$

$w = \pm 4$

critical pts

$+ \quad -4 \quad - \quad 0 \quad + \quad 4 \quad -$

$\boxed{\text{Increasing } (-\infty, -4), (0, 4) \quad \text{Decreasing } (-4, 0), (4, \infty)}$

10) ~~$f(t) = -\frac{1}{6}t^{\frac{7}{3}} + \frac{14}{3}t^{\frac{1}{3}} + 1$~~

~~$f'(t) = \frac{-7}{18}t$~~

For each problem, find the x-coordinates of all points of inflection and find the open intervals where the function is concave up and concave down.

11) $y = x^2 + 2x + 1$

$y' = 2x + 2$

$y'' = 2$

Concave up: $(-\infty, \infty)$

Concave down: none

No inflection points.

12) $y = x^5 - 3x^3 + 3$

$y' = 5x^4 - 9x^2$

$y'' = 20x^3 - 18x = 0$

$2x(10x^2 - 9) = 0$

$2x = 0 \qquad 10x^2 - 9 = 0$

$10x^2 = 9$

$x^2 = \frac{9}{10}$

Inflection pts: $x = 0 \quad x = \pm\frac{3\sqrt{10}}{10}$

concave up $\left(-\frac{3\sqrt{10}}{10}, 0\right)\left(\frac{3\sqrt{10}}{10}, \infty\right)$

concave down $\left(-\infty, -\frac{3\sqrt{10}}{10}\right)\left(0, \frac{3\sqrt{10}}{10}\right)$

13) ~~$y = \frac{12}{x^2 + 4}$~~

14) ~~$y = -\frac{1}{5}(x-4)^{\frac{5}{3}} - 2(x-4)^{\frac{2}{3}} - 2$~~

$y = -2\sin(2x) \quad [-\pi, \pi]$

$y' = -2\cos(2x) \cdot 2$

$= -4\cos(2x)$

$y'' = -4 \cdot -\sin(2x) \cdot 2$

$= 8\sin(2x) = 0$

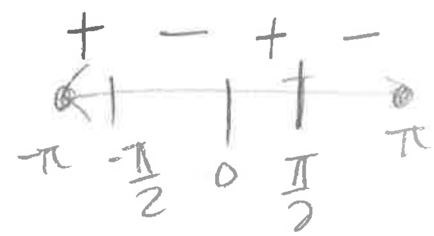
$\sin(2x) = 0$

$2x = 0 + 2\pi n$

$2x = \pi + 2\pi n$

$x = 0 + \pi n$

$x = \frac{\pi}{2} + \pi n$



inflection points:

$-\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

Concave up: $\left(-\pi, -\frac{\pi}{2}\right)\left(0, \frac{\pi}{2}\right)$

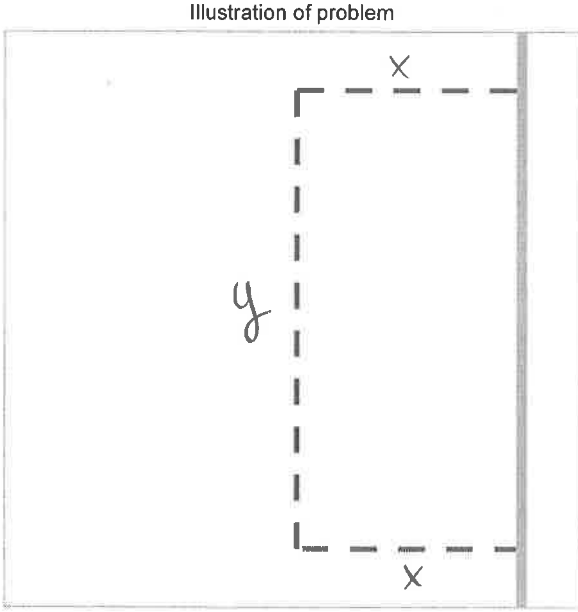
Concave down: $\left(-\frac{\pi}{2}, 0\right)\left(\frac{\pi}{2}, \pi\right)$

15) $y = 2\cos(x);\ [-\pi, \pi]$

Solve each optimization problem.

16) A farmer wants to construct a rectangular pigpen using 500 ft of fencing. The pen will be built next to an existing stone wall, so only three sides of fencing need to be constructed to enclose the pen. What dimensions should the farmer use to construct the pen with the largest possible area?

Illustration of problem



Constraint:

$2x + y = 500$

$y = 500 - 2x$

Maximize $A = xy$

$A = x(500 - 2x)$

$A = 500x - 2x^2$

$A' = 500 - 4x = 0$

$500 = 4x$

$125 = x$

$2(125) + y = 500$

$250 + y = 500$

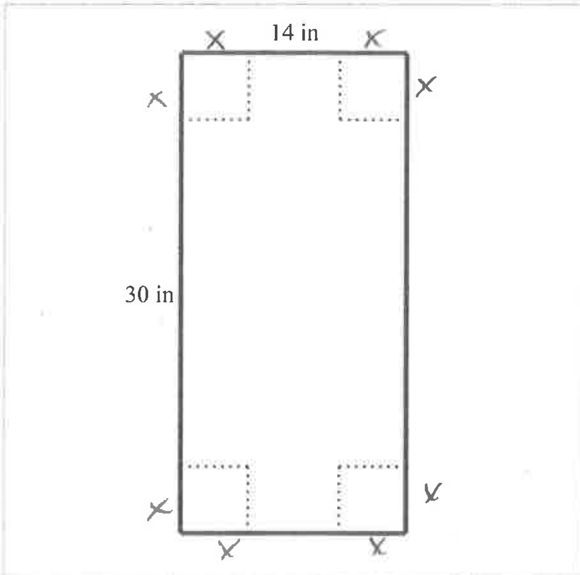
$y = 250$

Dimensions:

125 ft x 250 ft

17) A supermarket employee wants to construct an open-top box from a 14 by 30 in piece of cardboard. To do this, the employee plans to cut out squares of equal size from the four corners so the four sides can be bent upwards. What size should the squares be in order to create a box with the largest possible volume?

Illustration of problem



Maximize

$V = lwh$

$V = (30-2x)(14-2x)x$

$V = (4x^2 - 88x + 420)x$

$V = 4x^3 - 88x^2 + 420x$

$V' = 12x^2 - 176x + 420 = 0$

$4(3x^2 - 44x + 105) = 0$

$4(x-3)(3x-35) = 0$

$x = 3 \quad \cancel{x = \frac{35}{3}}$

$\boxed{\text{Sides of square} = 3\text{in}}$

18) We want to construct a cylindrical can with a bottom but no top that will have a volume of 65 in^3. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.

Constraint

$V = 65in^3$

$V = \pi r^2 h$

$65 = \pi r^2 h$

$h = \frac{65}{\pi r^2}$

Minimize

$M = \pi r^2 + 2\pi r h$

$M = \pi r^2 + 2\cancel{\pi} r\left(\frac{65}{\cancel{\pi} r^2}\right)$

$M = \pi r^2 + 130r^{-1}$

$M' = 2\pi r - 130r^{-2} = 0$

$2\pi r = \frac{130}{r^2}$

$2\pi r^3 = 130$

$r^3 = \frac{65}{\pi}$

$r = \sqrt[3]{\frac{65}{\pi}} \approx 2.745287471$

$h = \frac{65}{\pi(2.745287471)^2}$

$\approx$

Radius should be $\sqrt[3]{\frac{65}{\pi}} \approx 2.745287$in

Height should be approximately 2.745287471 in

19) The production costs, in dollars, per week of producing x widgets is given by $C(x) = 65000 + 4x + 0.2x^2 - 0.00002x^3$ and the demand function for widgets is given by $p(x) = 5000 - 0.5x$. What is the marginal cost, marginal revenue, and marginal profit when x=2000?

$C(x) = 65000 + 4x + 0.2x^2 - 0.00002x^3$

$C'(x) = 4 + 0.4x - 0.00006x^2$

$C'(2000) = 4 + 0.4(2000) - 0.00006(2000)^2 = 564$

The marginal cost of the 2001st widget is \$564.

$R(x) = x(p(x)) = x(5000 - 0.5x) = 5000x - 0.5x^2$

$R'(x) = 5000 - x$

$R'(2000) = 5000 - 2000 = 3000$

The marginal revenue of the 2001st widget is \$3000.

$P(x) = 5000x - 0.5x^2 + (-65000 - 4x - 0.2x^2 + 0.00002x^3)$

$= -65000 + 4996x - 0.7x^2 + 0.00002x^3$

$P'(x) = 4996 - 1.4x + 0.00006x^2$

$P'(2000) = 4996 - 1.4(2000) + 0.00006(2000)^2 = 2436$

The marginal profit of the 2001st widget is \$2436

Evaluate each limit using L'Hôpital's Rule.

17) $\lim_{x\to 0} \frac{\tan(x)}{5x} = \frac{0}{0}$

$\lim_{x\to 0} \frac{\sec^2(x)}{5} = \boxed{\frac{1}{5}}$

18) $\lim_{x\to\infty} 5x \cdot e^{-x} = \frac{5x}{e^x} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{5}{e^x} = \frac{5}{\infty} = \boxed{0}$

19) $\lim_{x\to\infty} \frac{2x}{\ln x} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{2}{\frac{1}{x}} = 2x = \boxed{\infty}$

20) $\lim_{x\to 0^+} 3 \cdot (\tan x)^{\sin x}$

$y = 3 \cdot \tan x^{\sin x}$

$\frac{1}{3} y = \tan x^{\sin x}$

$\ln \frac{1}{3} y = \ln \tan x^{\sin x}$

$\lim_{x\to 0^+} \boxed{\ln \frac{1}{3} y} = \sin x \ln \tan x = 0 \cdot \text{undefined}$

$= \frac{\ln \tan x}{\frac{1}{\sin x}} = \frac{\text{undefined}}{\text{undefined}}$

make this csc x

$\lim_{x\to 0^+} \boxed{\ln \frac{1}{3} y} = \frac{\frac{\sec^2 x}{\tan x}}{-\csc x \cot x} = \frac{\frac{1}{\cos^2 x} \cdot \frac{\cos x}{\sin x}}{-\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}}$

$= \frac{1}{\sin x \cos x} \cdot \frac{-\sin^2 x}{\cos x} = \frac{-\sin x}{\cos^2 x} = \frac{0}{1} = 0$

$e^{\ln \frac{1}{3} y} = e^0$

$\frac{1}{3} y = 1$

$y = 3$

21) $\lim_{x\to 0} \frac{2x^2}{e^x - 1 - x} = \frac{0}{0}$

$\lim_{x\to 0} \frac{4x}{e^x - 1} = \frac{0}{0}$

$\lim_{x\to 0} \frac{4}{e^x} = \frac{4}{1} = 4$