$m_{tan} = \frac{f(x)-f(a)}{x-a}$

Rates of Change and Tangent Lines

$f(-3)=3(-3+2)^2=3$

1. For the function $f(x)=3(x+2)^2$ and the point P given by $x=-3$ answer each of the following questions. $m_{PQ} = \frac{3(x+2)^2-3}{x-(-3)} = \frac{3(x+2)^2-3}{x+3}$

x	m_{PQ}
-3.5	-7.5
-3.1	-6.3
-3.01	-6.03
-3.001	-6.003
-3.0001	-6.0003

x	m_{PQ}
-2.5	-4.5
-2.9	-5.7
-2.99	-5.97
-2.999	-5.997
-2.9999	-5.9997

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) -3.5	**(ii)** -3.1	**(iii)** -3.01	**(iv)** -3.001	**(v)** -3.0001
(vi) -2.5	**(vii)** -2.9	**(viii)** -2.99	**(ix)** -2.999	**(x)** -2.9999

(b) Use the information from **(a)** to estimate the slope of the tangent line to $f(x)$ at $x=-3$ and write down the equation of the tangent line. $M=-6$ $y=mx+b$ $3=(-6)(-3)+b$ $3=18+b$ $-15=b$ $y=-6x-15$

2. For the function $g(x)=\sqrt{4x+8}$ and the point P given by $x=2$ answer each of the following questions. $M_{PQ}=\frac{\sqrt{4x+8}-4}{x-2}$ $f(2)=4$

use calculator Tableset

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

$M=.5$

(b) Use the information from **(a)** to estimate the slope of the tangent line to $g(x)$ at $x=2$ and write down the equation of the tangent line. $y=mx+b$ $y=\frac{1}{2}x+3$ $4=(.5)(2)+b$ $4=1+b$ $3=b$

3. For the function $W(x)=\ln(1+x^4)$ and the point P given by $x=1$ answer each of the following questions. $M_{PQ}=\frac{\ln(1+x^4)-\ln(2)}{x-1}$ $f(x)=\ln(2)$

Use Calculator

(a) For the points Q given by the following values of x compute (accurate to at least 8 decimal places) the slope, m_{PQ}, of the secant line through points P and Q.

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

$M=2$

(b) Use the information from **(a)** to estimate the slope of the tangent line to $W(x)$ at $x=1$ and write down the equation of the tangent line. $\ln(2)=(2)(1)+b$ $\ln(2)-2=b$ $y=2x+\ln(2)-2$

ARC - average rate of change

 $t(0.25)=3$

4. The volume of air in a balloon is given by $V(t)=\dfrac{6}{4t+1}$ answer each of the following questions. $ARC = \dfrac{\frac{6}{4t+1}-3}{t-0.25}$

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the volume of air in the balloon between $t=0.25$ and the following values of t. $M=-6$

(i) 1	**(ii)** 0.5	**(iii)** 0.251	**(iv)** 0.2501	**(v)** 0.25001
(vi) 0	**(vii)** 0.1	**(viii)** 0.249	**(ix)** 0.2499	**(x)** 0.24999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the volume of air in the balloon at $t=0.25$. At $t=0.25$, IRC $=-6$

$P(5)=10+\sin(0)=10$

5. The population (in hundreds) of fish in a pond is given by $P(t)=2t+\sin(2t-10)$ answer each of the following questions. $ARC = \dfrac{2t+\sin(2t-10)-10}{t-5}$

(a) Compute (accurate to at least 8 decimal places) the average rate of change of the population of fish between $t=5$ and the following values of t. Make sure your calculator is set to radians for the computations.

(i) 5.5	**(ii)** 5.1	**(iii)** 5.01	**(iv)** 5.001	**(v)** 5.0001
(vi) 4.5	**(vii)** 4.9	**(viii)** 4.99	**(ix)** 4.999	**(x)** 4.9999

(b) Use the information from **(a)** to estimate the instantaneous rate of change of the population of the fish at $t=5$. IRC $=400$

$s(2)=\cos^2(0)=\frac{1}{2}[1+\cos(2(0))]=\frac{1}{2}(1+1)=1$

6. The position of an object is given by $s(t)=\cos^2\left(\dfrac{3x-6}{2}\right)$ answer each of the following questions. $AV = \dfrac{\cos^2\left(\frac{3x-6}{2}\right)-1}{t-2}$

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t=2$ and the following values of t. Make sure your calculator is set to radians for the computations.

AV = average velocity

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t=2$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero). IRC $=0$, object is not moving

7. The position of an object is given by $s(t)=(8-x)(x+6)^{\frac{3}{2}}$. Note that a negative position here simply means that the position is to the left of the "zero position" and is perfectly acceptable.

Answer each of the following questions. $S(10) = (8-10)(10+6)^{3/2} = -2(64) = -128$

(a) Compute (accurate to at least 8 decimal places) the average velocity of the object between $t = 10$ and the following values of t. AV=

(i) 10.5	**(ii)** 10.1	**(iii)** 10.01	**(iv)** 10.001	**(v)** 10.0001
(vi) 9.5	**(vii)** 9.9	**(viii)** 9.99	**(ix)** 9.999	**(x)** 9.9999

(b) Use the information from **(a)** to estimate the instantaneous velocity of the object at $t = 10$ and determine if the object is moving to the right (*i.e.* the instantaneous velocity is positive), moving to the left (*i.e.* the instantaneous velocity is negative), or not moving (*i.e.* the instantaneous velocity is zero). $AV = \dfrac{(8-x)(x+6)^{3/2} + 128}{x-10}$ $IV = -76$

The Limit

1. For the function $f(x) = \dfrac{8 - x^3}{x^2 - 4}$ answer each of the following questions.

(a) Evaluate the function the following values of x compute (accurate to at least 8 decimal places).

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the value of $\lim\limits_{x \to 2} \dfrac{8 - x^3}{x^2 - 4}$. $= -3$

2. For the function $R(t) = \dfrac{2 - \sqrt{t^2 + 3}}{t + 1}$ answer each of the following questions.

(a) Evaluate the function the following values of t compute (accurate to at least 8 decimal places).

(i) -0.5	**(ii)** -0.9	**(iii)** -0.99	**(iv)** -0.999	**(v)** -0.9999
(vi) -1.5	**(vii)** -1.1	**(viii)** -1.01	**(ix)** -1.001	**(x)** -1.0001

(b) Use the information from **(a)** to estimate the value of $\lim\limits_{t \to -1} \dfrac{2 - \sqrt{t^2 + 3}}{t + 1}$. $= \dfrac{1}{2}$

3. For the function $g(\theta) = \dfrac{\sin(7\theta)}{\theta}$ answer each of the following questions.

(a) Evaluate the function the following values of θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 0.5 **(ii)** 0.1 **(iii)** 0.01 **(iv)** 0.001 **(v)** 0.0001
(vi) -0.5 **(vii)** -0.1 **(viii)** -0.01 **(ix)** -0.001 **(x)** -0.0001

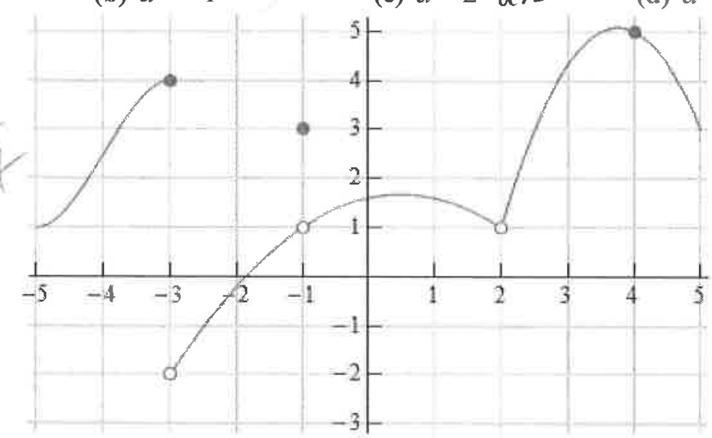
(b) Use the information from **(a)** to estimate the value of $\lim\limits_{\theta \to 0} \dfrac{\sin(7\theta)}{\theta}$. $=7$

4. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -3$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 4$

$f(-1)=3$ $\lim\limits_{a \to -1} = 1$ $f(2) = DNE$ $\lim\limits_{a \to 2} = 1$ $f(4)=5$ $\lim\limits_{a \to 4} = 5$

$f(-3)=4$
~~$\lim\limits_{a \to -3^-} = -3$~~
~~$\lim\limits_{a \to -3^+} = -2$~~ not yet
$\lim\limits_{a \to 3} = DNE$

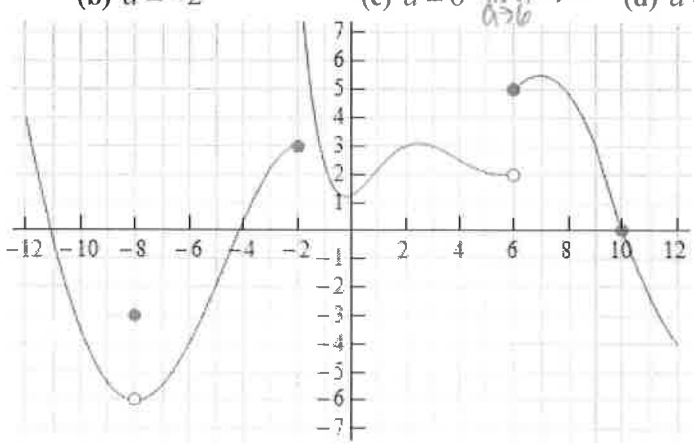


5. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -8$ **(b)** $a = -2$ **(c)** $a = 6$ **(d)** $a = 10$

$f(-2)=3$ $\lim\limits_{a \to -2} = DNE$ $f(6)=5$ $\lim\limits_{a \to 6} = DNE$ $f(10)=0$ $\lim\limits_{a \to 10} = 0$

$f(-8) = -3$
$\lim\limits_{a \to 8} = -6$



6. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$

f(-2) = DNE
lim = DNE
a→2

(b) $a = -1$

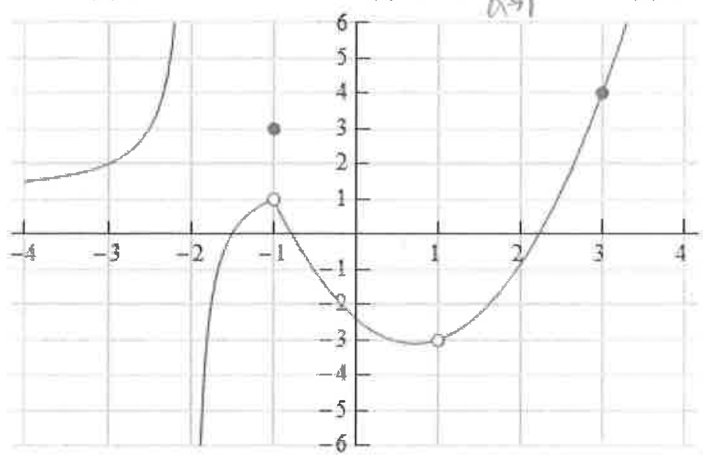
f(-1) = 3 lim = 1
a→-1

(c) $a = 1$

f(1) = DNE
lim = -3
a→1

(d) $a = 3$

f(3) = 4
lim = 4
a→3



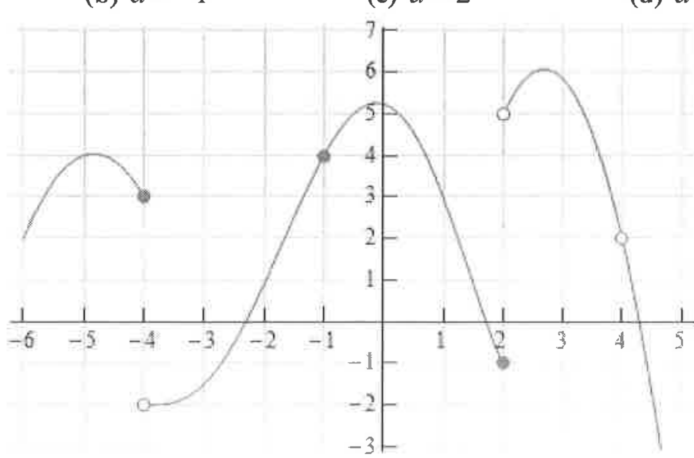
One-Sided Limits

1. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim\limits_{x \to a^-} f(x)$, $\lim\limits_{x \to a^+} f(x)$, and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -4$ (b) $a = -1$ (c) $a = 2$ (d) $a = 4$

(a) f(-4) = 3
lim = 3
x→-4⁻
lim = -2
x→-4⁺
lim = DNE
x→-4

(b) f(-1) = 4
lim = 4
x→-1⁻
lim = 4
x→-1⁺
lim = 4
x→-1

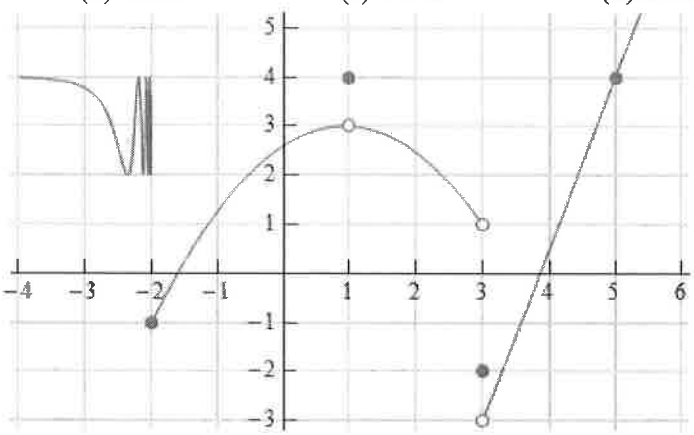


(c) f(2) = -1
lim = -1
x→2⁻
lim = 5
x→2⁺
lim = DNE
x→2

(d) f(4) = DNE
lim = 2
x→4⁻
lim = 2
x→4⁺
lim = 2
x→4

2. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$, $\lim\limits_{x\to a^-} f(x)$, $\lim\limits_{x\to a^+} f(x)$, and $\lim\limits_{x\to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a=-2$ **(b)** $a=1$ **(c)** $a=3$ **(d)** $a=5$

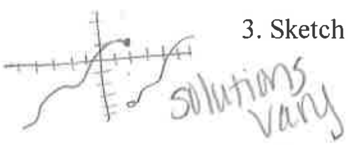


(a) $f(-2) = -1$
$\lim\limits_{x\to -2^-} = \text{DNE}$
$\lim\limits_{x\to -2^+} = -1$
$\lim\limits_{x\to -2} = \text{DNE}$

(b) $f(1) = 4$
$\lim\limits_{x\to 1^-} = 3$
$\lim\limits_{x\to 1^+} = 3$
$\lim\limits_{x\to 1} = 3$

(c) $f(3) = -2$
$\lim\limits_{x\to 3^-} = 1$
$\lim\limits_{x\to 3^+} = -3$
$\lim\limits_{x\to 3} = \text{DNE}$

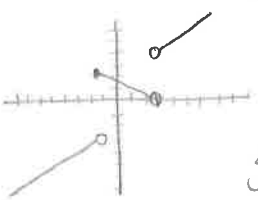
(d) $f(5) = 4$
$\lim\limits_{x\to 5^-} = 4$
$\lim\limits_{x\to 5^+} = 4$
$\lim\limits_{x\to 5} = 4$



3. Sketch a graph of a function that satisfies each of the following conditions.

$\lim\limits_{x\to 2^-} f(x) = 1$ $\lim\limits_{x\to 2^+} f(x) = -4$ $f(2) = 1$

solutions vary



4. Sketch a graph of a function that satisfies each of the following conditions.

$\lim\limits_{x\to 3^-} f(x) = 0$ $\lim\limits_{x\to 3^+} f(x) = 4$ $f(3)$ does not exist

$\lim\limits_{x\to -1} f(x) = -3$ $f(-1) = 2$

solutions vary

Limit Properties

1. Given $\lim\limits_{x\to 8} f(x) = -9$, $\lim\limits_{x\to 8} g(x) = 2$ and $\lim\limits_{x\to 8} h(x) = 4$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

$2(-9) - 12(4) = -66$

(a) $\lim\limits_{x\to 8}\left[2f(x) - 12h(x)\right]$

(b) $\lim\limits_{x\to 8}\left[3h(x) - 6\right]$

$3(4) - 6 = 6$

(c) $\lim\limits_{x\to 8}\left[g(x)h(x) - f(x)\right]$

$(2)(4) - (-9) = 17$

(d) $\lim\limits_{x\to 8}\left[f(x) - g(x) + h(x)\right]$

$-9 - 2 + 4 = -7$

2. Given $\lim_{x\to-4} f(x)=1$, $\lim_{x\to-4} g(x)=10$ and $\lim_{x\to-4} h(x)=-7$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x\to-4}\left[\frac{f(x)}{g(x)}-\frac{h(x)}{f(x)}\right]$ $\frac{1}{10}+\frac{+7}{1}=\frac{71}{10}$

(b) $\lim_{x\to-4}\left[f(x)g(x)h(x)\right]$ $1\cdot10\cdot-7=-70$

(c) $\lim_{x\to-4}\left[\frac{1}{h(x)}+\frac{3-f(x)}{g(x)+h(x)}\right]$ $\frac{1}{-7}+\frac{3-1}{10+-7}$ $=\frac{11}{21}$

(d) $\lim_{x\to-4}\left[2h(x)-\frac{1}{\underbrace{h(x)+7f(x)}_{=0}}\right]$ DNE

3. Given $\lim_{x\to0} f(x)=6$, $\lim_{x\to0} g(x)=-4$ and $\lim_{x\to0} h(x)=-1$ use the limit properties given in this section to compute each of the following limits. If it is not possible to compute any of the limits clearly explain why not.

(a) $\lim_{x\to0}\left[f(x)+h(x)\right]^3$ $(6+-1)^3=125$

(b) $\lim_{x\to0}\sqrt{g(x)h(x)}$ $\sqrt{(-4)(-1)}=2$

(c) $\lim_{x\to0}\sqrt[3]{11+\left[g(x)\right]^2}$ $\sqrt[3]{11+(-4)^2}=$ $\sqrt[3]{11+16}$ $\sqrt[3]{27}=3$

(d) $\lim_{x\to0}\sqrt{\frac{f(x)}{h(x)-g(x)}}$ $=\sqrt{\frac{6}{-1+4}}=\sqrt{2}$

For each of the following limits use the limit properties given in this section to compute the limit. At each step clearly indicate the property being used. If it is not possible to compute any of the limits clearly explain why not.

4. $\lim_{t\to-2}\left(14-6t+t^3\right)$ $=$ $\lim_{t\to-2}14-\lim_{t\to-2}6t+\lim_{t\to-2}t^3$ Property 2 / $\lim_{t\to-2}14-6\lim_{t\to-2}t+\lim_{t\to-2}t^3$ Property 1

$14-6(-2)+(-2)^3$ Properties 7, 8, 9 $=\boxed{18}$

5. $\lim_{x\to6}\left(3x^2+7x-16\right)$ $\lim_{x\to6}3x^2+\lim_{x\to6}7x-\lim_{x\to6}16$ Property 2 / $3\lim_{x\to6}x^2+7\lim_{x\to6}x-\lim_{x\to6}16$ Property 1

$3(6^2)+7(6)-16=\boxed{134}$ Properties 7, 8, 9

6. $\lim_{w\to3}\frac{w^2-8w}{4-7w}$ $\frac{\lim_{x\to3}w^2-8w}{\lim_{x\to3}4-7w}$ (P. 4) $\frac{\lim_{x\to3}w^2-8\lim_{x\to3}w}{\lim 4-7\lim_{x\to3}w}$ (P. 2 & 1) $\frac{3^2-8(3)}{4-7(3)}=\boxed{\frac{15}{17}}$ (P. 7, 8, 9)

7. $\lim_{x\to-5}\frac{x+7}{x^2+3x-10}$ $\frac{\lim_{x\to-5}x+7}{\lim_{x\to-5}x^2+3x-10}$ (P4) $\frac{\lim_{x\to-5}x+\lim_{x\to-5}7}{\lim_{x\to-5}x^2+3\lim_{x\to-5}x-\lim_{x\to-5}10}$ (P2 & 1) $\leftarrow \lim_{x\to-5}=0$ $\boxed{\text{DNE}}$

8. $\lim_{z\to0}\sqrt{z^2+6}$ $\sqrt{\lim_{z\to0}z^2+6}$ (P. 6) $\sqrt{\lim_{z\to0}z^2+\lim_{z\to0}6}$ (P2) $\sqrt{0^2+6}$ (P. 7 & 9) $=\boxed{\sqrt{6}}$

9. $\lim\limits_{x\to 10}\left(4x+\sqrt[3]{x-2}\right)$ $\lim\limits_{x\to 10} 4x + \lim\limits_{x\to 10}\sqrt[3]{x-2}$ (P2) $\lim\limits_{x\to 10} 4x + \sqrt[3]{\lim\limits_{x\to 10} x - \lim\limits_{x\to 10} 2}$ (P6+2)

$4\lim\limits_{x\to 10} x + \sqrt[3]{\lim\limits_{x\to 10} x - \lim\limits_{x\to 10} 2}$ (P1) $4(10)+\sqrt[3]{10-2} = \boxed{42}$ (P 7&8)

Computing Limits

For problems 1 – 9 evaluate the limit, if it exists.

1. $\lim\limits_{x\to 2}\left(8-3x+12x^2\right) = 8-3(2)+12(2)^2 = \boxed{50}$

2. $\lim\limits_{t\to -3}\dfrac{6+4t}{t^2+1} = \dfrac{6+4(-3)}{(-3)^2+1} = \boxed{-\dfrac{3}{5}}$

3. $\lim\limits_{x\to -5}\dfrac{x^2-25}{x^2+2x-15} = \dfrac{(-5)^2-25}{(-5)^2+2(-5)-15} = \dfrac{0}{0}$ $\dfrac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}(x-3)} = \dfrac{x-5}{x-3} = \dfrac{-5-5}{-5+3} = \boxed{\dfrac{5}{4}}$

4. $\lim\limits_{z\to 8}\dfrac{2z^2-17z+8}{8-z}$ $\dfrac{0}{0}$ $\dfrac{(2z-1)\cancel{(z-8)}}{-\cancel{(z-8)}} = -(2z-1) = -(2(8)-1) = \boxed{-15}$

5. $\lim\limits_{y\to 7}\dfrac{y^2-4y-21}{3y^2-17y-28}$ $\dfrac{0}{0}$ $\dfrac{\cancel{(y-7)}(y+3)}{(3y+4)\cancel{(y-7)}} = \dfrac{y+3}{3y+4} = \dfrac{7+3}{3(7)+4} = \boxed{\dfrac{2}{5}}$

6. $\lim\limits_{h\to 0}\dfrac{(6+h)^2-36}{h}$ $\dfrac{0}{0}$ $\dfrac{36+12h+h^2-36}{h} = \dfrac{\cancel{h}(12+h)}{\cancel{h}} = 12+0 = \boxed{12}$

7. $\lim\limits_{z\to 4}\dfrac{\sqrt{z}-2}{z-4}$ $\dfrac{0}{0}$ $\dfrac{(\sqrt{z}-2)(\sqrt{z}+2)}{(z-4)(\sqrt{z}+2)} = \dfrac{\cancel{(z-4)}}{\cancel{(z-4)}(\sqrt{z}+2)} = \dfrac{1}{\sqrt{z}+2} = \dfrac{1}{\sqrt{4}+2} = \boxed{\dfrac{1}{4}}$

8. $\lim\limits_{x\to -3}\dfrac{\sqrt{2x+22}-4}{x+3}$ $\dfrac{0}{0}$ $\dfrac{(\sqrt{2x+22}-4)(\sqrt{2x+22}+4)}{(x+3)(\sqrt{2x+22}+4)} = \dfrac{2x+22-16}{(x+3)(\sqrt{2x+22}+4)} = \dfrac{2x+6}{(x+3)(\sqrt{2x+22}+4)}$

$\dfrac{2\cancel{(x+3)}}{\cancel{(x+3)}(\sqrt{2x+22}+4)} = \dfrac{2}{\sqrt{2(-3)+22}+4} = \boxed{\dfrac{1}{4}}$

9. $\lim\limits_{x\to 0}\dfrac{x}{3-\sqrt{x+9}}$ $\dfrac{0}{0}$ $\dfrac{(x)}{(3-\sqrt{x+9})}\dfrac{(3+\sqrt{x+9})}{(3+\sqrt{x+9})} = \dfrac{x(3+\sqrt{x+9})}{9-x-9} = \dfrac{\cancel{x}(3+\sqrt{x+9})}{-\cancel{x}} = -(3+\sqrt{0+9}) = \boxed{-6}$

10. Given the function

$$f(x)=\begin{cases}7-4x & x<1\\ x^2+2 & x\geq 1\end{cases}$$

Evaluate the following limits, if they exist.

  http://tutorial.math.lamar.edu/terms.aspx

(a) $\lim\limits_{x \to -6} f(x)$ $7-4(-6)=\boxed{31}$

(b) $\lim\limits_{x \to 1} f(x)$ $\lim\limits_{x \to 1^-}(7-4x)=7-4(1)=3$ $\lim\limits_{x \to 1^+}(x^2+2)=1^2+2=3$ $\lim\limits_{x \to 1} f(x)=\boxed{3}$

11. Given

$$h(z)=\begin{cases} 6z & z \le -4 \\ 1-9z & z > -4 \end{cases}$$

Evaluate the following limits, if they exist.

(a) $\lim\limits_{z \to 7} h(z)$ $1-9(7)=\boxed{-62}$

(b) $\lim\limits_{z \to -4} h(z)$ $\lim\limits_{z \to -4^-} 6(-4)=-24$ $\lim\limits_{z \to -4^+} 1-9(-4)=37$ $\lim\limits_{z \to -4} \boxed{DNE}$

$|p|=\begin{cases} p & p \ge 0 \\ -p & p < 0 \end{cases}$

so $|x-5|=\begin{cases} x-5 & x \ge 5 \\ -(x-5) & x<5 \end{cases}$

For problems 12 & 13 evaluate the limit, if it exists.

12. $\lim\limits_{x \to 5}(10+|x-5|)$

$\lim\limits_{x \to 5^-}(10-(x-5))=\lim\limits_{x \to 5^-}(15-x)=10$

$\lim\limits_{x \to 5^+}(10+(x-5))=\lim\limits_{x \to 5^+}(5+x)=10$

$\lim\limits_{x \to 5}(10+|x-5|)=\boxed{10}$

plug in also works, but the absolute value is a piecewise function so we should evaluate from both sides

13. $\lim\limits_{t \to -1} \dfrac{t+1}{|t+1|}$

$\lim\limits_{t \to -1^-} \dfrac{t+1}{-(t+1)}=-1$

$\lim\limits_{t \to -1^+} \dfrac{t+1}{t+1}=1$

$\lim\limits_{t \to -1} \dfrac{t+1}{|t+1|}=\boxed{DNE}$

14. Given that $7x \le f(x) \le 3x^2+2$ for all x determine the value of $\lim\limits_{x \to 2} f(x)$.

$\lim\limits_{x \to 2} 7x=14$ $\lim\limits_{x \to 2} 3x^2+2=14$ $\lim\limits_{x \to 2} f(x)=\boxed{14}$

15. Use the Squeeze Theorem to determine the value of $\lim\limits_{x \to 0} x^4 \sin\left(\dfrac{\pi}{x}\right)$.

$-1 \le \sin\left(\frac{\pi}{x}\right) \le 1 \Rightarrow -x^4 \le x^4 \sin\left(\frac{\pi}{x}\right) \le x^4$

$\lim\limits_{x \to 0} -x^4=0$ $\lim\limits_{x \to 0} x^4=0$

$\lim\limits_{x \to 0} x^4 \sin\left(\frac{\pi}{x}\right)=\boxed{0}$

Infinite Limits

For problems 1 – 6 evaluate the indicated limits, if they exist.

1. For $f(x)=\dfrac{9}{(x-3)^5}$ evaluate,

$-$ $+$
3

(a) $\lim\limits_{x \to 3^-} f(x)=-\infty$ (b) $\lim\limits_{x \to 3^+} f(x)=+\infty$ (c) $\lim\limits_{x \to 3} f(x)=$ DNE

2. For $h(t)=\dfrac{2t}{6+t}$ evaluate,

$+$ $-$
-6

(a) $\lim\limits_{t \to -6^-} h(t)=\infty$ (b) $\lim\limits_{t \to -6^+} h(t)=-\infty$ (c) $\lim\limits_{t \to -6} h(t)=$ DNE

3. For $g(z)=\dfrac{z+3}{(z+1)^2}$ evaluate,

$+$ $+$
-1

(a) $\lim\limits_{z\to -1^-} g(z) = \infty$ **(b)** $\lim\limits_{z\to -1^+} g(z) = \infty$ **(c)** $\lim\limits_{z\to -1} g(z) = \infty$

4. For $g(x) = \dfrac{x+7}{x^2-4}$ evaluate,

$-$ $+$ 2

(a) $\lim\limits_{x\to 2^-} g(x) = -\infty$ **(b)** $\lim\limits_{x\to 2^+} g(x) = \infty$ **(c)** $\lim\limits_{x\to 2} g(x) = \text{DNE}$

Don't be scared of $(-x)$

5. For $h(x) = \ln(-x)$ evaluate, If x is negative, the argument is + this is the same as $\lim\limits_{x\to 0^+} \ln(x) = -\infty$

(a) $\lim\limits_{x\to 0^-} h(x) = -\infty$ **(b)** $\lim\limits_{x\to 0^+} h(x) = \text{DNE}$ **(c)** $\lim\limits_{x\to 0} h(x) = \text{DNE}$

6. For $R(y) = \tan(y)$ evaluate,

$+$ $-$ $3\pi/2$

(a) $\lim\limits_{y\to \frac{3\pi}{2}^-} R(y) = \infty$ **(b)** $\lim\limits_{y\to \frac{3\pi}{2}^+} R(y) = -\infty$ **(c)** $\lim\limits_{y\to \frac{3\pi}{2}} R(y) = \text{DNE}$

For problems 7 & 8 find all the vertical asymptotes of the given function.

7. $f(x) = \dfrac{7x}{(10-3x)^4}$

$10-3x=0$

$x = \frac{10}{3}$

$\lim\limits_{x\to 10/3} f(x) = \dfrac{70/3}{0}$

$+$ $+$ $\frac{70}{3}$

vertical asymptote $x = \frac{10}{3}$

8. $g(x) = \dfrac{-8}{(x+5)(x-9)}$

$x=-5$

$x=9$

$-$ $+$ -5

$\lim\limits_{x\to -5^-} g(x) = -\infty$

$\lim\limits_{x\to -5^+} g(x) = \infty$

$+$ $-$ 9

$\lim\limits_{x\to 9^-} g(x) = \infty$

$\lim\limits_{x\to 9^+} g(x) = -\infty$

vertical asymptotes $x=5$ $x=9$

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Limits At Infinity, Part I

1. For $f(x) = 4x^7 - 18x^3 + 9$ evaluate each of the following limits.

(a) $\lim\limits_{x\to -\infty} f(x) = x^7\left(4 - \frac{18}{x^4} + \frac{9}{x^7}\right) = -\infty(4) = -\infty$ **(b)** $\lim\limits_{x\to \infty} f(x) = \infty(4) = \infty$

2. For $h(t) = \sqrt[3]{t} + 12t - 2t^2$ evaluate each of the following limits.

t^2 $(-\infty)^2$ ∞

(a) $\lim\limits_{t\to -\infty} h(t) = t^2\left(\frac{1}{t^{5/3}} + \frac{12}{t} - 2\right)$ $(\infty)(-2) = -\infty$ **(b)** $\lim\limits_{t\to \infty} h(t) = (\infty)(-2) = -\infty$

For problems 3 – 10 answer each of the following questions.

(a) Evaluate $\lim\limits_{x\to -\infty} f(x)$.

(b) Evaluate $\lim\limits_{x\to \infty} f(x)$.

(c) Write down the equation(s) of any horizontal asymptotes for the function.

3. $f(x)=\dfrac{8-4x^2}{9x^2+5x}$ (a) $\dfrac{\cancel{x^2}\left(\frac{8}{x^2}-4\right)}{\cancel{x^2}\left(9+\frac{5}{x}\right)}=\dfrac{-4}{9}$ (b) $-\dfrac{4}{9}$ (c) $y=-4/9$

4. $f(x)=\dfrac{3x^7-4x^2+1}{5-10x^2}$ (a) $\dfrac{\cancel{x^2}\left(3x^5-4+\frac{1}{x^2}\right)}{\cancel{x^2}\left(\frac{5}{x^2}-10\right)}=\dfrac{-\infty}{-10}=\infty$ (b) $\dfrac{\infty}{-10}=-\infty$ (c) NONE

5. $f(x)=\dfrac{20x^4-7x^3}{2x+9x^2+5x^4}$ (a) $\dfrac{\cancel{x^4}\left(20-\frac{7}{x}\right)}{\cancel{x^4}\left(\frac{2}{x^3}+\frac{9}{x^2}+5\right)}=\dfrac{20}{5}=4$ (b) 4 (c) $y=4$

6. $f(x)=\dfrac{x^3-2x+11}{3-6x^5}$ (a) $\dfrac{\cancel{x^5}\left(\frac{1}{x^2}-\frac{2}{x^4}+\frac{11}{x^5}\right)}{\cancel{x^5}\left(\frac{3}{x^5}-6\right)}=\dfrac{0}{-6}=0$ (b) 0 (c) $y=0$

7. $f(x)=\dfrac{x^6-x^4+x^2-1}{7x^6+4x^3+10}$ (a) $\dfrac{x^6\left(1-\frac{1}{x^2}+\frac{1}{x^4}-\frac{1}{x^6}\right)}{x^6\left(7+\frac{4}{x^3}+\frac{10}{x^6}\right)}=\dfrac{1}{7}$ (b) $\dfrac{1}{7}$ (c) $y=1/7$

8. $f(x)=\dfrac{\sqrt{7+9x^2}}{1-2x}$ (a) $\dfrac{-\cancel{x}\sqrt{\frac{7}{x^2}+9}}{\cancel{x}\left(\frac{1}{x}-2\right)}=\dfrac{-\sqrt{9}}{-2}=\dfrac{3}{2}$ (b) $\dfrac{\sqrt{9}}{-2}=\dfrac{-3}{2}$ (c) $y=-3/2$, $y=3/2$

9. $f(x)=\dfrac{x+8}{\sqrt{2x^2+3}}$ (a) $\dfrac{x\left(1+\frac{8}{x}\right)}{-x\sqrt{2+\frac{3}{x^2}}}=\dfrac{1}{-\sqrt{2}}=\dfrac{-\sqrt{2}}{2}$ (b) $\dfrac{\sqrt{2}}{2}$ (c) $y=-\sqrt{2}/2$, $y=\sqrt{2}/2$

$\sqrt{x^4}=|x^2|=x^2$

10. $f(x)=\dfrac{8+x-4x^2}{\sqrt{6+x^2+7x^4}}$ (a) $\dfrac{x^2\left(\frac{8}{x^2}+\frac{1}{x}-4\right)}{x^2\sqrt{\frac{6}{x^4}+\frac{1}{x^2}+7}}=\dfrac{-4}{\sqrt{7}}=\dfrac{-4\sqrt{7}}{7}$ (b) $\dfrac{-4\sqrt{7}}{7}$ (c) $y=\dfrac{-4\sqrt{7}}{7}$

$\lim\limits_{x\to\infty} e^x=\infty$ $\lim\limits_{x\to-\infty} e^x=0$ $\lim\limits_{x\to\infty} e^{-x}=0$ $\lim\limits_{x\to-\infty} e^{-x}=\infty$

Limits At Infinity, Part II

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For problems 1 – 6 evaluate (a) $\lim\limits_{x\to-\infty} f(x)$ and (b) $\lim\limits_{x\to\infty} f(x)$.

1. $f(x)=\mathbf{e}^{8+2x-x^3}$ (a) $\lim\limits_{x\to-\infty}(8+2x-x^3)=x^3\left(\frac{8}{x^3}+\frac{2}{x^2}-1\right)=(-\infty)(-1)=\infty$ $\lim\limits_{x\to-\infty} e^{8+2x-x^3}=\boxed{\infty}$ (b) $\lim\limits_{x\to\infty} x^3\left(\frac{8}{x^3}+\frac{2}{x^2}-1\right)=(\infty)(-1)=-\infty$ $\lim\limits_{x\to\infty} e^{8+2x-x^3}=\boxed{0}$

2. $f(x)=\mathbf{e}^{\frac{6x^2+x}{5+3x}}$ (a) $\lim\limits_{x\to-\infty}\dfrac{6x^2+x}{5+3x}$ $\dfrac{\cancel{x}(6x+1)}{\cancel{x}\left(\frac{5}{x}+3\right)}=\dfrac{-\infty}{3}=-\infty$ $\lim\limits_{x\to-\infty} e^{\frac{6x^2+x}{5+3x}}=\boxed{0}$ (b) $\lim\limits_{x\to\infty}\left(\dfrac{6x^2+x}{5+3x}\right)=\dfrac{\infty}{3}=\infty$ so $\lim\limits_{x\to\infty} e^{\frac{6x^2+x}{5+3x}}=\boxed{\infty}$

3. $f(x)=2\mathbf{e}^{6x}-\mathbf{e}^{-7x}-10\mathbf{e}^{4x}$ (a) $\lim\limits_{x\to-\infty}\left(2e^{6x}-e^{-7x}-10e^{4x}\right)=0-\infty-0=\boxed{-\infty}$

(b) $\lim\limits_{x\to\infty}\left(2e^{6x}-e^{-7x}-10e^{4x}\right)=\infty-0-\infty$ $\lim\limits_{x\to\infty}\left(2e^{6x}\left(1-5e^{-2x}\right)-e^{-7x}\right)$ $(\infty(1)-0)=\boxed{\infty}$

$\lim\limits_{x\to\infty} e^x = \infty$ $\lim\limits_{x\to\infty} e^x = 0$ $\lim\limits_{x\to\infty} e^{-x} = 0$ $\lim\limits_{x\to-\infty} e^{-x} = \infty$

4. $f(x) = 3\mathbf{e}^{-x} - 8\mathbf{e}^{-5x} - \mathbf{e}^{10x}$ ⓐ $\infty - \infty - 0$ $\lim\limits_{x\to-\infty} \left(e^{-5x}(3e^{4x}-8) - e^{10x}\right) = \infty(-8) - 0 = \boxed{-\infty}$

ⓑ $0 - 0 - \infty = \boxed{-\infty}$

5. $f(x) = \dfrac{\mathbf{e}^{-3x} - 2\mathbf{e}^{8x}}{9\mathbf{e}^{8x} - 7\mathbf{e}^{-3x}}$ ⓐ $\dfrac{e^{-3x}(1-2e^{11x})}{e^{-3x}(9e^{11x}-7)} = \dfrac{1-0}{-7} = \boxed{\dfrac{1}{-7}}$ ⓑ $\dfrac{e^{8x}(e^{-11x}-2)}{e^{8x}(9-7^{-11x})} = \dfrac{0-2}{9-0} = \boxed{\dfrac{-2}{9}}$

causing ∞ for $\lim\limits_{x\to-\infty}$

6. $f(x) = \dfrac{\mathbf{e}^{-7x} - 2\mathbf{e}^{3x} - \mathbf{e}^{x}}{\mathbf{e}^{-x} + 16\mathbf{e}^{10x} + 2\mathbf{e}^{-4x}}$ ⓐ $\dfrac{e^{-4x}(e^{-3x} - 2^{7x} - e^{5x})}{e^{-4x}(e^{3x} + 16e^{14x} + 2)} = \dfrac{\infty - 0 - 0}{0 + 0 + 2} = \boxed{\infty}$ ⓑ $\dfrac{e^{10x}(e^{-17x} - 2e^{-7x} - e^{-9x})}{e^{10x}(e^{-11x} + 16 + 2e^{-14x})} = \dfrac{0-0-0}{0+16+0} = \boxed{0}$

$\lim\limits_{x\to0^+} \ln x = -\infty$

$\lim\limits_{x\to\infty} \ln x = \infty$

For problems 7 – 12 evaluate the given limit.

7. $\lim\limits_{t\to-\infty} \ln\left(4 - 9t - t^3\right)$ $\lim\limits_{t\to-\infty} (4 - 9t - t^3) = -\infty(0+0-1) = +\infty$ $\lim\limits_{t\to-\infty} \ln(4 - 9t - t^3) = \boxed{\infty}$

8. $\lim\limits_{z\to-\infty} \ln\left(\dfrac{3z^4 - 8}{2 + z^2}\right)$ $\lim\limits_{z\to-\infty} \left(\dfrac{3z^4-8}{2+z^2}\right) = \dfrac{\infty - 0}{0+1} = \infty$ $\lim\limits_{z\to-\infty} \ln\left(\dfrac{3z^4-8}{2+z^2}\right) = \boxed{\infty}$

$\lim\limits_{x\to-\infty} \tan^{-1} x = \dfrac{-\pi}{2}$

9. $\lim\limits_{x\to\infty} \ln\left(\dfrac{11 + 8x}{x^3 + 7x}\right)$ $\lim\limits_{x\to\infty} \left(\dfrac{11+8x}{x^3+7x}\right) = \dfrac{0+0}{1+0} = 0$ $\lim\limits_{x\to\infty} \ln\left(\dfrac{11+8x}{x^3+7x}\right) = \boxed{-\infty}$

$\lim\limits_{x\to\infty} \tan^{-1} x = \dfrac{\pi}{2}$

10. $\lim\limits_{x\to-\infty} \tan^{-1}\left(7 - x + 3x^5\right)$ $\lim\limits_{x\to-\infty} (7 - x + 3x^5) = -\infty(3) = -\infty$ $\lim\limits_{x\to-\infty} \tan^{-1}(7 - x + 3x^5) = \boxed{\dfrac{-\pi}{2}}$

11. $\lim\limits_{t\to\infty} \tan^{-1}\left(\dfrac{4 + 7t}{2 - t}\right)$ $\lim\limits_{t\to\infty} \left(\dfrac{4+7t}{2-t}\right) = \dfrac{0+7}{0-1} = -7$ $\lim\limits_{x\to\infty} \tan^{-1}\left(\dfrac{4+7t}{2-t}\right) = \boxed{\tan^{-1}(-7)}$

12. $\lim\limits_{w\to\infty} \tan^{-1}\left(\dfrac{3w^2 - 9w^4}{4w - w^3}\right)$ $\lim\limits_{w\to\infty} \left(\dfrac{3w^2 - 9w^4}{4w - w^3}\right) = \dfrac{0-\infty}{0-1} = \infty$ $\lim\limits_{w\to\infty} \tan^{-1}\left(\dfrac{3w^2 - 9w^4}{4w - w^3}\right) = \boxed{\dfrac{\pi}{2}}$

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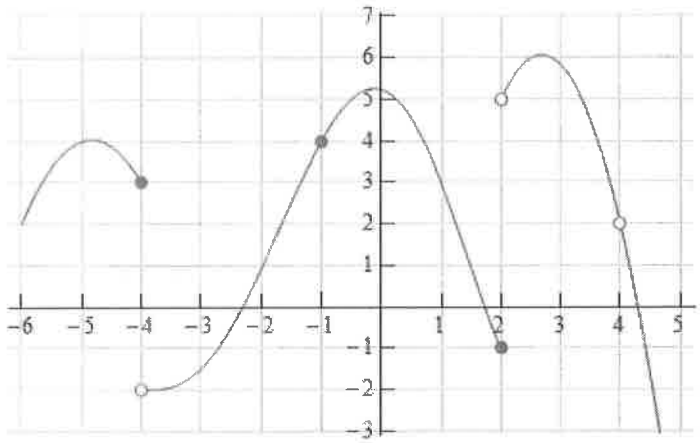
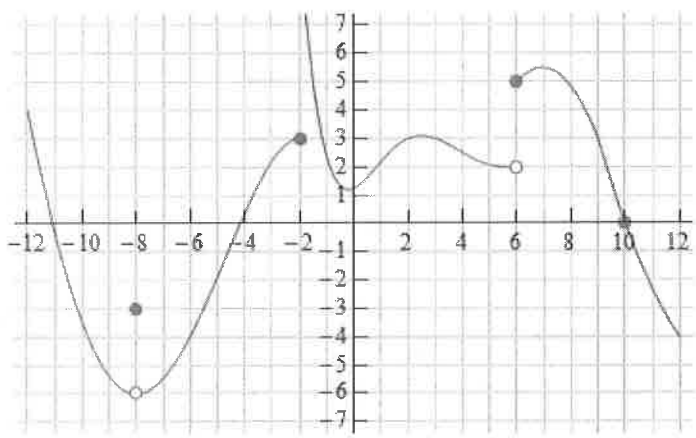
Continuity

1. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



x	reason
-4	lim DNE
2	lim DNE
4	f(4) DNE

2. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



x	reason
-8	f(-8) DNE
-2	lim DNE
6	lim DNE

For problems 3 – 7 using only Properties 1 – 9 from the Limit Properties section, one-sided limit properties (if needed) and the definition of continuity determine if the given function is continuous or discontinuous at the indicated points. Instead, determine if there is a hole or vertical asymptote

3. $f(x) = \dfrac{4x+5}{9-3x}$ vertical asymptote @ x=3

(a) $x=-1$, **(b)** $x=0$, **(c)** $x=3$?

4. $g(z) = \dfrac{6}{z^2-3z-10}$ $(z-5)(z+2)$ vertical asymptotes at z=-2 and z=5

(a) $z=-2$, **(b)** $z=0$, **(c)** $z=5$?

5. $g(x) = \begin{cases} 2x & x<6 \\ x-1 & x \geq 6 \end{cases}$ $2(6)=12$ $6-1=5$ hole at x=6

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(a) $x = 4$, **(b)** $x = 6$?

6. $h(t) = \begin{cases} t^2 & t < -2 \quad (-2)^2 = 4 \\ t+6 & t \geq -2 \quad -2+6=4 \end{cases}$ no discontinuity

(a) $t = -2$, **(b)** $t = 10$?

7. $g(x) = \begin{cases} 1-3x & x < -6 \quad 1-3(-6)=19 \\ 7 & x = -6 \\ x^3 & -6 < x < 1 \quad 1^3 = 1 \\ 1 & x = 1 \\ 2-x & x > 1 \quad 2-1=1 \end{cases}$ hole at $x=-6$

(a) $x = -6$, **(b)** $x = 1$?

For problems 8 – 12 determine where the given function is discontinuous.

8. $f(x) = \dfrac{x^2 - 9}{3x^2 + 2x - 8}$ $\dfrac{(x-3)(x+3)}{(3x-4)(x+2)}$ v. asymptote at $x = 4/3$, $x=-2$
h. asymptote at $y = 1/3$

9. $R(t) = \dfrac{8t}{t^2 - 9t - 1}$ Can't factor $\dfrac{9 \pm \sqrt{81-4(1)(1)}}{2(1)} = x = \dfrac{9 \pm \sqrt{85}}{2}$ v. asymptote at $x = \dfrac{9 \pm \sqrt{85}}{2}$
h asymptote at $y=0$

10. $h(z) = \dfrac{1}{2 - 4\cos(3z)}$ denom zero when $\cos(3z) = 1/2$
$\cos^{-1}(1/2) = 3z$
$3z = \pi/3$
$3z = \frac{5\pi}{3}$
v asymptote at $z = \pi/9 + \frac{2\pi n}{3}$
$z = 5\pi/9 + \frac{2\pi n}{3}$

11. $y(x) = \dfrac{x}{7 - \mathbf{e}^{2x+3}}$ $\ln e^{2x+3} = \ln 7$
$2x+3 = \ln 7$
$2x = \ln 7 - 3$
$x = \frac{\ln 7 - 3}{2}$
v. asymptote at $x = \frac{\ln(7)-3}{2}$

12. $g(x) = \tan(2x)$ discontinuous at $\tan(\pi/2) + 2\pi n$ & $\tan(3\pi/2) + 2\pi n$
$2x = \pi/2 + 2\pi n$
$2x = 3\pi/2 + 2\pi n$
v. asymptotes at $x = \frac{\pi}{4} + \pi n$ and $x = \frac{3\pi}{4} + \pi n$

For problems 13 – 15 use the Intermediate Value Theorem to show that the given equation has at least one solution in the indicated interval. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval.

13. $25 - 8x^2 - x^3 = 0$ on $[-2, 4]$ continuous (polynomial) $M=0$
$f(-2) = 1$ $f(4) = -167$ $f(4) = -167 < 0 < 1 = f(-2)$
$\therefore f(c) = 0$

14. $w^2 - 4\ln(5w+2) = 0$ on $[0, 4]$ continuous (+) $M=0$
$f(0) = -2.7726$ $f(4) = 3.6358$ $f(0) < 0 < f(4)$
$\therefore f(c) = 0$

15. $4t + 10\mathbf{e}^t - \mathbf{e}^{2t} = 0$ on $[1, 3]$ continuous $M=0$
$f(1) = 23.7938$ $f(3) = -190.5734$ $f(3) < 0 < f(1)$
$\therefore f(c) = 0$