38-52

The Definition of the Limit

Problems for this section have not yet been written.

Derivatives

Introduction

Here are a set of practice problems for the Derivatives chapter of my Calculus I notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a number of places on the site.

10. If you'd like a pdf document containing the solutions go to the note page for the section you'd like solutions for and select the download solutions link from there. Or,

11. Go to the download page for the site http://tutorial.math.lamar.edu/download.aspx and select the section you'd like solutions for and a link will be provided there.

12. If you'd like to view the solutions on the web or solutions to an individual problem you can go to the problem set web page, select the problem you want the solution for. At this point I do not provide pdf versions of individual solutions, but for a particular problem you can select "Printable View" from the "Solution Pane Options" to get a printable version.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have practice problems written for them.

$(x+h)(x^2+2hx+h^2)$

$x^3+2hx^2+xh^2$
$+hx^2+xh^2+h^3$

⑦ $\lim_{h\to 0} \frac{(x+h)^3-2(x+h)^2+(x+h)-1-x^3+2x^2-x+1}{h}$

$= \frac{\cancel{x^3}+3hx^2+2xh^2+h^3-\cancel{2x^2}-4hx-2h^2+\cancel{x}+h-\cancel{1}-\cancel{x^3}+\cancel{2x^2}-\cancel{x}+\cancel{1}}{h}$

$= \frac{h(3x^2+2xh+h^2-4x-2h+1)}{h} = 3x^2-4x+1$

The Definition of the Derivative

Use the definition of the derivative to find the derivative of the following functions.

1. $f(x)=6$ $\lim_{h\to 0} \frac{6-6}{h} = \frac{0}{h} = \frac{0}{0}$ on either side $\frac{0}{\#}=0$ $=0$

2. $V(t)=3-14t$ $\lim_{h\to 0} \frac{3-14(t+h)-(3-14t)}{h} = \frac{3-14t-14h-3+14t}{h} = \frac{-14\cancel{h}}{\cancel{h}} = -14$

3. $g(x)=x^2$ $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \frac{x^2+2hx+h^2-x^2}{h} = \frac{h(2x+h)}{h} = 2x+h = 2x$

4. $Q(t)=10+5t-t^2$ $\lim_{h\to 0} \frac{10+5(t+h)-(t+h)^2-(10+5t-t^2)}{h} = \frac{10+5t+5h-\cancel{t^2}-2ht-h^2-10-5t+\cancel{t^2}}{h} = \frac{h(5-2t-h)}{h} = 5-2t$

5. $W(z)=4z^2-9z$ $\lim_{h\to 0} \frac{4(z+h)^2-9(z+h)-(4z^2-9z)}{h} = \frac{4(z^2+2hz+h^2)-9z-9h-4z^2+9z}{h} = \frac{4z^2+8hz+4h^2-9z-9h-4z^2+9z}{h}$
$= \frac{h(8z+4h-9)}{h} = 8z-9$

6. $f(x)=2x^3-1$ $\lim_{h\to 0} \frac{2(x+h)^3-1-(2x^3-1)}{h} = \frac{2(x+h)(x^2+2hx+h^2)-1-2x^3+1}{h} = \frac{2(x^3+3hx^2+3xh^2+h^3)-1-2x^3+1}{h}$
$= \frac{\cancel{2x^3}+6hx^2+6xh^2-2h^3-\cancel{2x^3}}{h} = \frac{h(6x^2+6xh-2h^2)}{h} = 6x^2$

7. $g(x)=x^3-2x^2+x-1$ See above

8. $R(z)=\frac{5}{z}$ $\lim_{h\to 0} \frac{\frac{5z}{z(z+h)}-\frac{5(z+h)}{z(z+h)}}{h} = \frac{\frac{5z-5z-5h}{z(z+h)}}{h} = \frac{-5h}{z(z+h)}\cdot\frac{1}{h} = \frac{-5}{z(z+h)} = \frac{-5}{z^2}$

9. $V(t)=\frac{t+1}{t+4}$ $\lim_{h\to 0} \frac{\frac{(t+h)+1}{(t+h)+4}\frac{(t+4)}{(t+4)}-\frac{t+1}{t+4}\frac{(t+h+4)}{(t+h+4)}}{h}$ $\frac{t^2+ht+t+4t+4h+4-(t^2+ht+4t+1t+h+4)}{h(t+h+4)(t+4)} = \frac{3\cancel{h}}{\cancel{h}(t+h+4)(t+4)} = \frac{3}{(t+4)^2}$

10. $Z(t)=\sqrt{3t-4}$ $\lim_{h\to 0} \frac{\sqrt{3(t+h)-4}-\sqrt{3t-4}}{h}\cdot\frac{(\sqrt{3(t+h)-4}+\sqrt{3t-4})}{(\sqrt{3(t+h)-4}+\sqrt{3t-4})} = \frac{3t+3h-4-3t+4}{h(\sqrt{3(t+h)-4}+\sqrt{3t-4})} = \frac{3}{2\sqrt{3t-4}}$

11. $f(x)=\sqrt{1-9x}$ $\lim_{h\to 0} \frac{(\sqrt{1-9(x+h)}-\sqrt{1-9x})}{h}\cdot\frac{(\sqrt{1-9(x+h)}+\sqrt{1-9x})}{(\sqrt{1-9(x+h)}+\sqrt{1-9x})} = \frac{1-9(x+h)-(1-9x)}{h(\sqrt{1-9(x+h)}+\sqrt{1-9x})}$

$= \frac{1-9x-9h-1+9x}{h(\sqrt{1-9(x+h)}+\sqrt{1-9x})} = \frac{-9h}{h(\sqrt{1-9(x+h)}+\sqrt{1-9x})} = \frac{-9}{2\sqrt{1-9x}}$

CALC

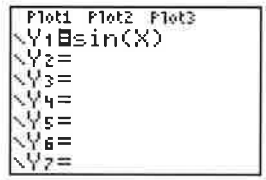
Getting Started: Drawing a Tangent Line

Getting Started is a fast-paced introduction. Read the chapter for details.

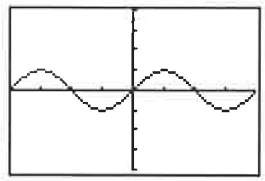
Suppose you want to find the equation of the tangent line at X = √2/2 for the function Y = sinX.

Before you begin, select **Radian** and **Func** mode from the mode screen, if necessary.

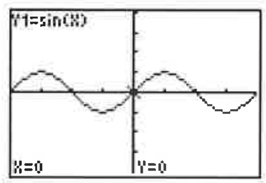
1. Press [Y=] to display the Y= editor. Press [SIN] [X,T,Θ,*n*] [)] to store **sin(X)** in **Y1**.



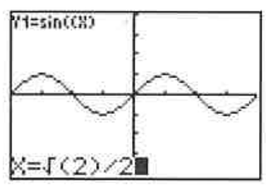
2. Press [ZOOM] **7** to select **7:ZTrig**, which graphs the equation in the Zoom Trig window.



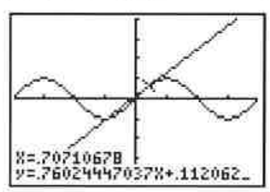
3. Press [2nd] [DRAW] **5** to select **5:Tangent(**. The tangent instruction is initiated.



4. Press [2nd] [√] **2** [)] [÷] **2**.



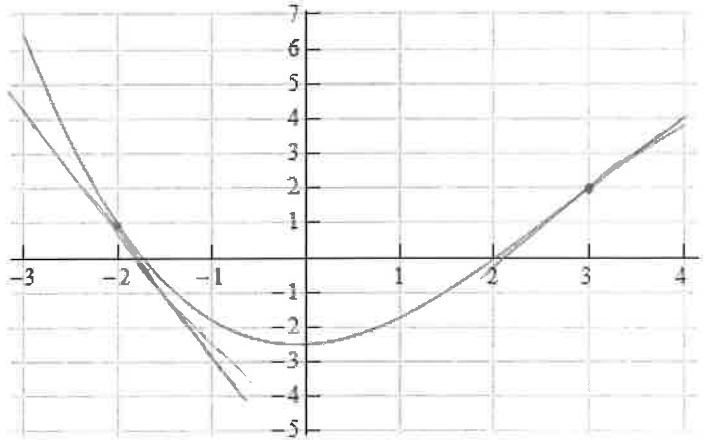
5. Press [ENTER]. The tangent line is drawn; the **X** value and the tangent-line equation are displayed on the graph.



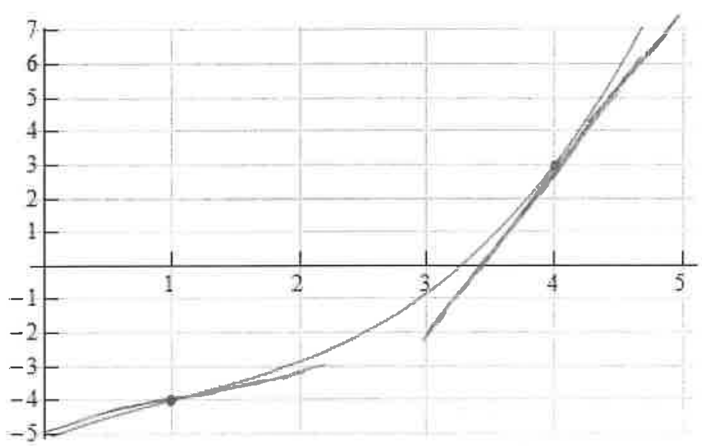
Interpretations of the Derivative

For problems 1 and 2 use the graph of the function, $f(x)$, estimate the value of $f'(a)$ for the given values of a.

1. **(a)** $a=-2$ **(b)** $a=3$ $f'(-2)=-4$ $f'(3)=2$

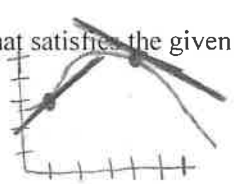


2. **(a)** $a=1$ **(b)** $a=4$ $f'(1)=1$ $f'(4)=5$

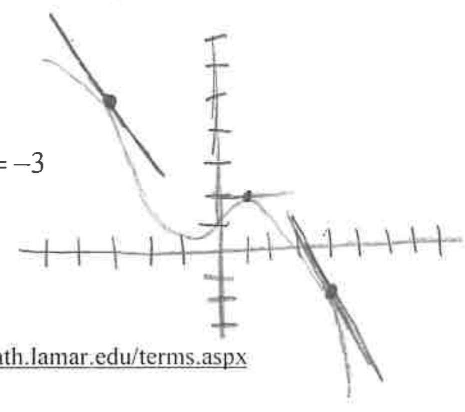


For problems 3 and 4 sketch the graph of a function that satisfies the given conditions.

3. $f(1)=3$, $f'(1)=1$, $f(4)=5$, $f'(4)=-2$



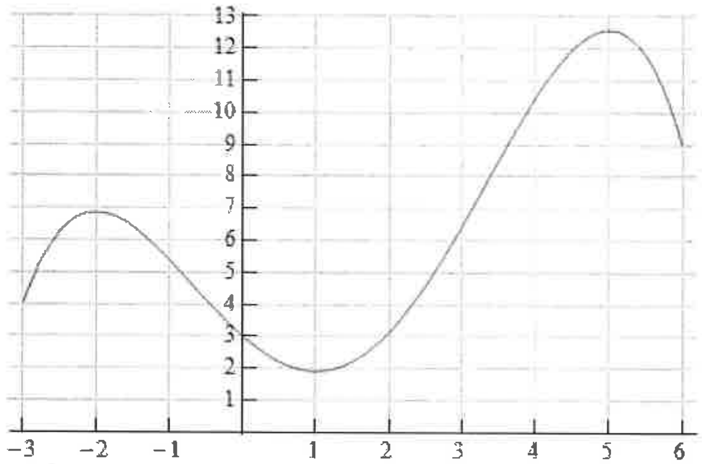
4. $f(-3)=5$, $f'(-3)=-2$, $f(1)=2$, $f'(1)=0$, $f(4)=-2$, $f'(4)=-3$



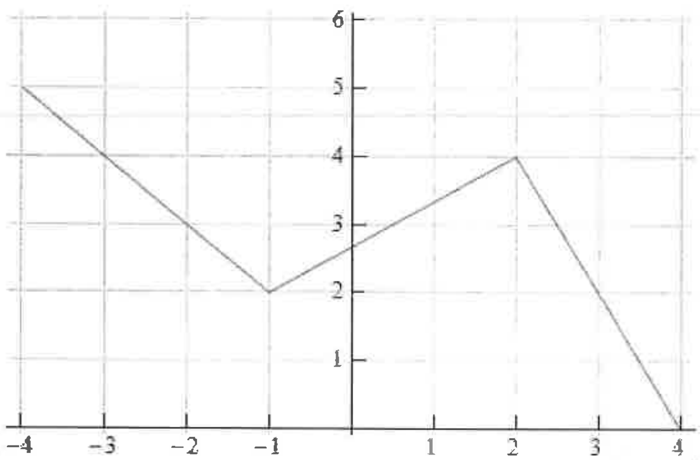
For problems 5 and 6 the graph of a function, $f(x)$, is given. Use this to sketch the graph of the derivative, $f'(x)$.

5.

$f'(-2)=0$
$f'(1)=0$
$f'(5)=0$



6.



See previous #5

7. Answer the following questions about the function $W(z)=4z^2-9z$.

$\frac{4(z+h)^2-9(z+h)-(4z^2-9z)}{h}$

$\frac{4(z^2+2zh+h^2)-9z-9h-4z^2+9z}{h}$

$\frac{4z^2+8zh+4h^2-9z-9h-4z^2+9z}{h}$

$\frac{h(8z+4h-9)}{h}=8z-9$

(a) Is the function increasing or decreasing at $z=-1$? $8(-1)-9=-17$ decrease

(b) Is the function increasing or decreasing at $z=2$? $8(2)-9=7$ increase

(c) Does the function ever stop changing? If yes, at what value(s) of z does the function stop changing? $8z-9=0$ $z=9/8$ yes

8. What is the equation of the tangent line to $f(x)=3-14x$ at $x=8$.

$3-14(x+h)-(3-14x)$

$\frac{3-14x-14h-3+14x}{h}=-14$

$f(8)=3-14(8)=-109$

$f'(8)=-14$

$y=-109+(-14)(x-8)$

$y=-14x+3$

$\cdot(t+4)$

$\frac{1}{h}\left(\frac{(t+h)+1}{(t+h)+4} - \frac{(t+1)}{t+4}\right)$ $(t+h+4)$ $(t+h+4)$

$(t+4)$

$\frac{1}{h}\left(\frac{\cancel{t^2}+\cancel{ht}+\cancel{t}+4t+4h+\cancel{4}-(\cancel{t^2}+\cancel{ht}+4t+t+h+4)}{(t+h+4)(t+4)}\right)$

see previous #9

9. The position of an object at any time t is given by $s(t)=\frac{t+1}{t+4}$. $\frac{1}{\cancel{h}}\cdot\frac{3\cancel{h}}{(t+h+4)(t+4)} = \frac{3}{(t+4)(t+4)}$

(a) Determine the velocity of the object at any time t. $s'(t)=\frac{3}{(t+4)^2}$

(b) Does the object ever stop moving? If yes, at what time(s) does the object stop moving? $\frac{3}{(t+4)^2}=0$ Not possible doesn't stop moving

see previous #8

10. What is the equation of the tangent line to $f(x)=\frac{5}{x}$ at $x=\frac{1}{2}$? $f'(x)=\frac{-5}{x^2}$ $f(\frac{1}{2})=10$ $f'(\frac{1}{2})=-20$ $y=10+(-20)(x-\frac{1}{2})$ $y=-20x+20$

see previous #7

11. Determine where, if anywhere, the function $g(x)=x^3-2x^2+x-1$ stops changing. $3x^2-4x+1=0$

$(x-1)(3x-1)=0$

$x=1$ $x=\frac{1}{3}$

see previous #10

12. Determine if the function $Z(t)=\sqrt{3t-4}$ increasing or decreasing at the given points. $Z'(t)=\frac{3}{2\sqrt{3t-4}}$

(a) $t=5$ $Z'(5)=\frac{3}{2\sqrt{3(5)-4}}=\frac{3}{2\sqrt{11}}$ + means increasing

(b) $t=10$ $Z'(10)=\frac{3}{2\sqrt{3(10)-4}}=\frac{3}{2\sqrt{26}}$ + means increasing

(c) $t=300$ $Z'(300)=\frac{3}{2\sqrt{3(300)-4}}=\frac{3}{2\sqrt{896}}$ + means increasing

13. Suppose that the volume of water in a tank for $0\le t\le 6$ is given by $Q(t)=10+5t-t^2$. see previous #4 $Q'(t)=5-2t$

(a) Is the volume of water increasing or decreasing at $t=0$? $Q'(0)=5-2(0)=5$ + means increasing

(b) Is the volume of water increasing or decreasing at $t=6$? $Q'(6)=5-2(6)=-7$ − means decreasing

(c) Does the volume of water ever stop changing? If yes, at what times(s) does the volume stop changing? Yes, it has to

$5-2t=0$

$-2t=-5$

$t=5/2$

Differentiation Formulas

For problems 1 – 12 find the derivative of the given function.

1. $f(x)=6x^3-9x+4$ $f'(x)=18x^2-9$

2. $y=2t^4-10t^2+13t$ $y'=8t^3-20t+13$

3. $g(z)=4z^7-3z^{-7}+9z$ $g'(z)=28z^6+\frac{21}{z^8}+9$

4. $h(y)=y^{-4}-9y^{-3}+8y^{-2}+12$ $h'(y)=-\frac{4}{y^5}+\frac{27}{y^4}-\frac{16}{y^3}$

5. $y=\sqrt{x}+8\sqrt[3]{x}-2\sqrt[4]{x}$ $y=x^{1/2}+8x^{1/3}-2x^{1/4}$ $y'=\frac{1}{2}x^{-1/2}+\frac{8}{3}x^{-2/3}-\frac{1}{2}x^{-3/4}=\frac{1}{2\sqrt{x}}+\frac{8}{3\sqrt[3]{x^2}}-\frac{1}{2\sqrt[4]{x^3}}$

$10x^{3/5} - x^{7/2} + 6x^{8/3} - 3$

6. $f(x)=10\sqrt[5]{x^3}-\sqrt{x^7}+6\sqrt[3]{x^8}-3$ $f'(x)=6x^{-2/5}-\frac{7}{2}x^{5/2}+16x^{5/3}=\frac{6}{\sqrt[5]{x^2}}-\frac{7\sqrt{x^5}}{2}+16\sqrt[3]{x^5}$

7. $f(t)=\frac{4}{t}-\frac{1}{6t^3}+\frac{8}{t^5}=4t^{-1}-\frac{1}{6}t^{-3}+8t^{-5}$ $f'(t)=-4t^{-2}+\frac{1}{2}t^{-4}-40t^{-6}=-\frac{4}{t^2}+\frac{1}{2t^4}-\frac{40}{t^6}$

8. $R(z)=\frac{6}{\sqrt{z^3}}+\frac{1}{8z^4}-\frac{1}{3z^{10}}=6z^{-3/2}+\frac{1}{8}z^{-4}-\frac{1}{3}z^{-10}$ $R'(z)=-9z^{-5/2}-\frac{1}{2}z^{-5}+\frac{10}{3}z^{-11}=\frac{-9}{z^2\sqrt{z}}-\frac{1}{2z^5}+\frac{10}{3z^{11}}$

9. $z=x(3x^2-9)=3x^3-9x$ $z'=9x^2-9$

10. $g(y)=(y-4)(2y+y^2)=2y^2+y^3-8y-4y^2=y^3-2y^2-8y$ $g'(y)=3y^2-4y-8$

11. $h(x)=\frac{4x^3-7x+8}{x}=4x^2-7+8x^{-1}$ $h'(x)=8x-8x^{-2}=8x-\frac{8}{x^2}$

12. $f(y)=\frac{y^5-5y^3+2y}{y^3}=y^2-5+2y^{-2}$ $f'(y)=2y-4y^{-3}=2y-\frac{4}{y^3}$

F 13. Determine where, if anywhere, the function $f(x)=x^3+9x^2-48x+2$ is not changing.

$f'(x)=3x^2+18x-48=0$ $3(x^2+6x-16)=0$ $3(x+8)(x-2)=0$ $x=-8$ $x=2$

Q 14. Determine where, if anywhere, the function $y=2z^4-z^3-3z^2$ is not changing.

$y'=8z^3-3z^2-6z$ $z(8z^2-3z-6)=0$ $x=\frac{3\pm\sqrt{9-4(8)(-6)}}{2(8)}=\frac{3\pm\sqrt{201}}{16}, 0$

15. Find the tangent line to $g(x)=\frac{16}{x}-4\sqrt{x}$ at $x=4$.

$g(x)=16x^{-1}-4x^{1/2}$ $g'(x)=-16x^{-2}-2x^{-1/2}$ $g(4)=-4$ $g'(4)=-2$ $y=-4+(-2)(x-4)$ $y=-2x+4$

16. Find the tangent line to $f(x)=7x^4+8x^{-6}+2x$ at $x=-1$.

$f'(x)=28x^3-48x^{-7}+2$ $f(-1)=13$ $f'(-1)=22$ $y=13+(22)(x+1)$ $y=22x+35$

17. The position of an object at any time t is given by $s(t)=3t^4-40t^3+126t^2-9$.

(a) Determine the velocity of the object at any time t. $s'(t)=12t^3-120t^2+252t$

(b) Does the object ever stop changing? $12t(t-3)(t-7)=0$ $t=3$, $t=7$, $t=0$

(c) When is the object moving to the right and when is the object moving to the left? time is not negative

$s'(1)=144$ + right $s'(4)=-144$ − left $s'(8)=480$ + right
Right $(0,3)\cup(7,\infty)$
Left $(3,7)$

(18) $h'(z)=120z^2-20z^3-20z^4$
$-20z^2(z-2)(z+3)$
$z=0$, $z=2$, $z=-3$
$h'(-4)=-1920$ $h'(-1)=120$ $h'(1)=80$ $h'(3)=-1080$
− −3 + 0 + 2 −

18. Determine where the function $h(z)=6+40z^3-5z^4-4z^5$ is increasing and decreasing.

increasing $(-3,0)\cup(0,2)$ decreasing $(-\infty,-3)\cup(2,\infty)$

19. Determine where the function $R(x)=(x+1)(x-2)^2$ is increasing and decreasing.

increasing $(-\infty,0)\cup(2,\infty)$ decreasing $(0,2)$

(19) $(x+1)(x^2-4x+4)$
$x^3-4x^2+4x+x^2-4x+4$
x^3-3x^2+4
$R'(x)=3x^2-6x$
$3x(x-2)$
$x=0$, $x=2$
$R'(-1)$ + $R'(1)$ − $R'(3)$ +
0 2

need slope of 4

Q 20. Determine where, if anywhere, the tangent line to $f(x) = x^3 - 5x^2 + x$ is parallel to the line $y = 4x + 23$.

$f'(x) = 3x^2 - 10x + 1$

$3x^2 - 10x + 1 = 4$

$3x^2 - 10x - 3 = 0$

$\frac{10 \pm \sqrt{100 - 4(3)(-3)}}{2(3)}$

$x = \frac{5 \pm \sqrt{34}}{3}$

Product and Quotient Rule

For problems 1 – 6 use the Product Rule or the Quotient Rule to find the derivative of the given function.

1. $f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$

$8t-1 \quad 3t^2 - 16t$

$f'(t) = (8t-1)(t^3 - 8t^2 + 12) + (4t^2 - t)(3t^2 - 16t) = 20t^4 - 132t^3 + 24t^2 + 96t - 12$

2. $y = \left(1 + \sqrt{x^3}\right)\left(x^{-3} - 2\sqrt[3]{x}\right)$

$\frac{3}{2}x^{1/2} \quad -3x^{-4} - \frac{2}{3}x^{-2/3}$

$y' = \left(\frac{3}{2}x^{1/2}\right)\left(x^{-3} - 2x^{1/3}\right) + \left(1 + x^{3/2}\right)\left(-3x^{-4} - \frac{2}{3}x^{-2/3}\right) = -3x^{-4} - \frac{3}{2}x^{-5/2} - \frac{2}{3}x^{-2/3} - \frac{11}{3}x^{5/6}$

3. $h(z) = (1 + 2z + 3z^2)(5z + 8z^2 - z^3)$

$2 + 6z \quad 5 + 16z - 3z^2$

$h'(z) = (2 + 6z)(5z + 8z^2 - z^3) + (1 + 2z + 3z^2)(5 + 16z - 3z^2) = 5 + 36z + 90z^2 + 88z^3 - 15z^4$

4. $g(x) = \frac{6x^2}{2 - x}$

$12x \quad -1$

$g'(x) = \frac{12x(2-x) - 6x^2(-1)}{(2-x)^2} = \frac{24x - 12x^2 + 6x^2}{(2-x)^2} = \frac{24x - 6x^2}{(2-x)^2}$

5. $R(w) = \frac{3w + w^4}{2w^2 + 1}$

$3 + 4w^3 \quad 4w$

$R'(w) = \frac{(3 + 4w^3)(2w^2 + 1) - (3w + w^4)(4w)}{(2w^2 + 1)^2} = \frac{4w^5 + 4w^3 - 6w^2 + 3}{(2w^2 + 1)^2}$

6. $f(x) = \frac{\sqrt{x} + 2x}{7x - 4x^2}$

$\frac{1}{2}x^{-1/2} + 2 \quad 7 - 8x$

$f'(x) = \frac{\left(\frac{1}{2}x^{-1/2} + 2\right)(7x - 4x^2) - (x^{1/2} + 2x)(7 - 8x)}{(7x - 4x^2)^2} = \frac{\frac{7}{2}x^{1/2} + \cancel{14x} - 8x^2 - 2x^{3/2} + (-7x^{1/2} - \cancel{14x} + 8x^{3/2} + 16x^2)}{(7x - 4x^2)^2}$

$= \frac{-\frac{7}{2}x^{1/2} + 8x^2 + 6x^{3/2}}{(7x - 4x^2)^2}$

7. If $f(2) = -8$, $f'(2) = 3$, $g(2) = 17$ and $g'(2) = -4$ determine the value of $(fg)'(2)$.

$3 \cdot 17 + -8 \cdot -4 = 83$

8. If $f(x) = x^3 g(x)$, $g(-7) = 2$, $g'(-7) = -9$ determine the value of $f'(-7)$.

$f'(x) = 3x^2(g(x)) + x^3(g'(x))$

$f'(-7) = 3x^2(2) + x^3(-9)$

$= 3(-7)^2(2) + (-7)^3(-9)$

$= 3381$

9. Find the equation of the tangent line to $f(x) = (1 + 12\sqrt{x})(4 - x^2)$ at $x = 9$.

$6x^{-1/2} \quad -2x$

$f'(x) = (6x^{-1/2})(4 - x^2) + (1 + 12x^{1/2})(-2x)$

$f'(9) = -820 \quad f(9) = -2849$

tan line $= -2849 - 820(x - 9)$

$y = -820x + 4531$

10. Determine where $f(x) = \frac{x - x^2}{1 + 8x^2}$ is increasing and decreasing.

$1 - 2x \quad 16x$

11. Determine where $V(t) = (4 - t^2)(1 + 5t^2)$ is increasing and decreasing.

$-2t \quad 10t$

(10) $f'(x) = \frac{(1-2x)(1+8x^2) - (x - x^2)(16x)}{(1+8x^2)^2} = \frac{1 - 2x - 8x^2}{(1+8x^2)^2} = 0$

$1 - 2x - 8x^2 = 0$

$-(4x - 1)(2x + 1) = 0$

$x = 1/4 \quad x = -1/2$

$f'(-1) \quad f'(0) \quad f'(1)$

$- \quad -1/2 \quad + \quad 1/4 \quad -$

increasing $(-1/2, 1/4)$ decreasing $(-\infty, -1/2) \cup (1/4, \infty)$

(11) $V'(t) = (-2t)(1 + 5t^2) + (4 - t^2)(10t) = 38t - 20t^3$

$= 2t(19 - 10t^2)$

$2t(19 - 10t^2) = 0$

$t = 0 \quad t = \pm\sqrt{\frac{19}{10}} \approx \pm 1.3784$

$V'(-2) \quad V'(-1) \quad V'(1) \quad V'(2)$

$+ \quad - \quad + \quad -$

$-1.3784 \quad 0 \quad 1.3784$

increasing $\left(-\infty, -\sqrt{\frac{19}{10}}\right) \cup \left(0, \sqrt{\frac{19}{10}}\right)$

decreasing $\left(-\sqrt{\frac{19}{10}}, 0\right) \cup \left(\sqrt{\frac{19}{10}}, \infty\right)$

Derivatives of Trig Functions

For problems 1 – 3 evaluate the given limit.

1. $\lim\limits_{z\to 0}\dfrac{\sin(10z)}{z}\cdot\dfrac{10}{10} = 10\lim\limits_{z\to 0}\dfrac{\sin(10z)}{(10z)} = 10\cdot 1 = \boxed{10}$

2. $\lim\limits_{\alpha\to 0}\dfrac{\sin(12\alpha)}{\sin(5\alpha)}\cdot\dfrac{12\alpha}{12\alpha}\cdot\dfrac{5\alpha}{5\alpha} = \dfrac{12\alpha}{5\alpha}\cdot\dfrac{\sin(12\alpha)}{12\alpha}\cdot\dfrac{5\alpha}{\sin(5\alpha)} = \boxed{\dfrac{12}{5}}$

3. $\lim\limits_{x\to 0}\dfrac{\cos(4x)-1}{x}\cdot\dfrac{4}{4} = 4\lim\limits_{x\to 0}\dfrac{\cos(4x)-1}{4x} = 4\cdot 0 = \boxed{0}$

For problems 4 – 10 differentiate the given function. $\dfrac{1}{\cos x}$ $\dfrac{\sin x}{\cos x}$

4. $f(x)=2\cos(x)-6\sec(x)+3$ $\boxed{f'(x) = -2\sin(x) - 6\sec(x)\tan(x)}$

5. $g(z)=10\tan(z)-2\cot(z)$ $\boxed{g'(z) = 10\sec^2(z) + 2\csc^2(z)}$

P 6. $f(w)=\tan(w)\sec(w)$ $f'(w) = (\sec^2(w))(\sec(w)) + (\tan(w))(\sec(w)\tan(w)) = \boxed{\sec^3(w) + \sec(w)\tan^2(w)}$
$\sec^2(w)$ $\sec(w)\tan(w)$

P 7. $h(t)-t^3-t^2\sin(t)$ $h'(t) = 3t^2 - [(2t)(\sin(t)) + (t^2)(\cos(t))] = 3t^2 - 2t\sin(t) - t^2\cos(t)$
$2t$ $\cos(t)$

P 8. $y=\cancel{6}+4\sqrt{x}\csc(x)$ $\dfrac{dy}{dx} = \left(\dfrac{2}{\sqrt{x}}\right)(\csc(x)) + (4\sqrt{x})(-\csc(x)\cot(x)) = \boxed{\dfrac{2\csc(x)}{\sqrt{x}} - 4\sqrt{x}\csc(x)\cot(x)}$
$\dfrac{2}{\sqrt{x}}$ $-\csc(x)\cot(x)$

Q 9. $R(t)=\dfrac{1}{2\sin(t)-4\cos(t)}$ 0 $R'(t) = \dfrac{0(2\sin(t)-4\cos(t)) - 1(2\cos(t)+4\sin(t))}{(2\sin(t)-4\cos(t))^2} = \boxed{\dfrac{-2\cos(t)-4\sin(t)}{(2\sin(t)-4\cos(t))^2}}$
$2\cos(t)+4\sin(t)$

Q 10. $Z(v)=\dfrac{v+\tan(v)}{1+\csc(v)}$ $1+\sec^2(v)$ $-\csc(v)\cot(v)$ $Z'(v) = \dfrac{(1+\sec^2(v))(1+\csc(v)) + (v+\tan(v))(+\csc(v)\cot(v))}{(1+\csc(v))^2}$ See add'l sheet!

11. Find the tangent line to $f(x)=\tan(x)+9\cos(x)$ at $x=\pi$. $f(\pi) = -9$ $f'(\pi) = 1$ tan line $y = -9 + 1(x-\pi)$ $\boxed{y = x - \pi - 9}$
$\sec^2(x) - 9\sin(x)$

12. The position of an object is given by $s(t)=2+7\cos(t)$ determine all the points where the object is not moving. $s'(t) = -7\sin(t)$ $-7\sin(t)=0$ $\sin(t)=0$ $t = 0+2\pi n$ and $t = \pi + 2\pi n$

13. Where in the range $[-2,7]$ is the function $f(x)=4\cos(x)-x$ is increasing and decreasing. $f'(x) = -4\sin(x) - 1$ See add'l sheet

$-4\sin(x) = +1$
$\sin(x) = -1/4$
$x = \sin^{-1}(-1/4) \approx -0.2527 \leftarrow$ no neg solutions
$2\pi - 0.2527 = 6.0305$
$\sin$ $\pi - a = \pi + 0.2527 = 3.3943$
$x = 3.3943 + 2\pi n$
$x = 6.0305 + 2\pi n$
$x = -0.2527$
3.3943
6.0305

Trig

#10

multiply negatives

$\cancel{+}(v+\tan v)(+\csc v\cot v)$

online → $$\frac{(1+\sec^2 v)(1+\csc v) + (v+\tan v)(\csc v\cot v)}{(1+\csc v)^2}$$

distribute

I prefer this

OK to go further →

don't have to

$$\frac{1+\sec^2 v+\boxed{\csc v}+\csc v\sec^2 v+v\csc v\cot v+\boxed{\csc v\,\cancel{\cot v\tan v}}}{(1+\csc v)^2}$$

these don't simplify

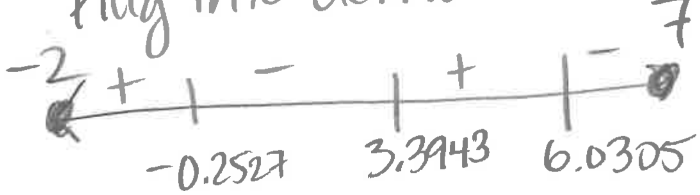
$\frac{1}{\sin v}\cdot\frac{1}{\cos^2 v}$ $\quad\frac{1}{\sin v}\cdot\frac{\cos v}{\sin v}$ $\quad\frac{1}{\sin v}\cdot\cancel{\frac{\cos v}{\sin v}\,\frac{\sin v}{\cos v}}$

or

→ $$\frac{1+\sec^2 v+2\csc v+\csc v\sec^2 v+v\csc v\cot v}{(1+\csc v)^2}$$

#13 continued

Plug into derivative



Increasing $(-\infty,-0.2527)\cup(3.3943,6.0305)$

Decreasing $(-0.2527,3.3943)\cup(6.0305,\infty)$

Derivatives of Exponential and Logarithm Functions

For problems 1 – 6 differentiate the given function.

1. $f(x) = 2\mathbf{e}^x - 8^x$ $\boxed{f'(x) = 2e^x - 8^x \ln(8)}$

2. $g(t) = 4\log_3(t) - \ln(t)$ $\boxed{g'(t) = \frac{4}{t\ln 3} - \frac{1}{t}}$

Q 3. $R(w) = 3^w \log(w)$ $\boxed{R'(w) = 3^w \ln(3) \log(w) + \frac{3^w}{w \ln 10}}$

Q 4. $y = z^5 - \mathbf{e}^z \ln(z)$ $y' = 5z^4 - \left(e^z \ln(z) + \frac{e^z}{z}\right) = \boxed{5z^4 - e^z \ln(z) + \frac{e^z}{z}}$

Q 5. $h(y) = \frac{y}{1 - \mathbf{e}^y}$ $h'(y) = \frac{1(1-e^y) - y(-e^y)}{(1-e^y)^2} = \boxed{\frac{1 - e^y + ye^y}{(1-e^y)^2}}$

Q 6. $f(t) = \frac{1 + 5t}{\ln(t)}$ $f'(t) = \frac{5\ln(t) - (1+5t)(1/t)}{[\ln(t)]^2} = \boxed{\frac{5\ln(t) - \frac{1}{t} - 5}{[\ln(t)]^2}}$

7. Find the tangent line to $f(x) = 7^x + 4\mathbf{e}^x$ at $x = 0$. $f(0) = 5$ $y = 5 + 5.9459x$
$f'(x) = 7^x \ln 7 + 4e^x$ $f'(0) = \ln(7) + 4 = 5.9459$

Q 8. Find the tangent line to $f(x) = \ln(x)\log_2(x)$ at $x = 2$. $f(2) = \ln(2)$ $y = \ln(2) + 1(x-2)$
$f'(x) = \frac{1}{x}(\log_2(x)) + \ln(x)\frac{1}{x\ln 2}$ $f'(2) = \frac{\log_2(2)}{2} + \frac{\ln(2)}{2\ln(2)} = 1$ $y = \ln(2) + x - 2$

$V'(t) = \frac{e^t - te^t}{(e^t)^2}$ Q 9. Determine if $V(t) = \frac{t}{\mathbf{e}^t}$ is increasing or decreasing at the following points.

Factor out e^t
$V'(t) = \frac{1-t}{e^t}$

(a) $t = -4$ + increasing (b) $t = 0$ + increasing (c) $t = 10$ − decreasing

Q 10. Determine if $G(z) = (z-6)\ln(z)$ is increasing or decreasing at the following points. 10 $G'(z) = \ln z + \frac{1}{z}(z-6)$

(a) $z = 1$ − decreasing (b) $z = 5$ + increasing (c) $z = 20$ + increasing

Derivatives of Inverse Trig Functions

For each of the following problems differentiate the given function.

1. $T(z) = 2\cos(z) + 6\cos^{-1}(z)$ $\boxed{T'(z) = -2\sin(z) - \frac{6}{\sqrt{1-z^2}}}$

2. $g(t) = \csc^{-1}(t) - 4\cot^{-1}(t)$ $\boxed{g'(t) = -\frac{1}{|t|\sqrt{t^2-1}} + \frac{4}{1+t^2}}$

3. $y = 5x^6 - \sec^{-1}(x)$ $\boxed{y' = 30x^5 - \frac{1}{|x|\sqrt{x^2-1}}}$

Q 4. $f(w) = \sin(w) + w^2\tan^{-1}(w)$ $\boxed{f'(w) = \cos(w) + 2w\tan^{-1}(w) + \frac{w^2}{1+w^2}}$

Q 5. $h(x) = \frac{\sin^{-1}(x)}{1+x}$ $\frac{1}{\sqrt{1-x^2}}$ 1 $h'(x) = \frac{\frac{1}{\sqrt{1-x^2}}(1+x) - \sin^{-1}(x)}{(1+x)^2}$ common denom $= \boxed{\frac{1+x-\sqrt{1-x^2}\sin^{-1}(x)}{(1+x)^2\sqrt{1-x^2}}}$

Derivatives of Hyperbolic Functions

For each of the following problems differentiate the given function.

1. $f(x) = \sinh(x) + 2\cosh(x) - \text{sech}(x)$ $\boxed{f'(x) = \cosh(x) + 2\sinh(x) + \text{sech}(x)\tanh(x)}$

Q 2. $R(t) = \tan(t) + t^2\,\text{csch}(t)$ $\boxed{R'(t) = \sec^2(t) + 2t\,\text{csch}(t) - t^2\,\text{csch}(x)\coth(x)}$

Q 3. $g(z) = \frac{z+1}{\tanh(z)}$ 1 $\text{sech}^2(x)$ $\boxed{g'(z) = \frac{\tanh(z) - (z+1)(\text{sech}^2(z))}{[\tanh(z)]^2}}$

Chain Rule

For problems 1 – 26 differentiate the given function.

1. $f(x) = (6x^2 + 7x)^4$ $4(6x^2+7x)^3(12x+7)$

2. $g(t) = (4t^2 - 3t + 2)^{-2}$ $-2(4t^2-3t+2)^{-3}(8t-3)$

3. $y = \sqrt[3]{1-8z}$ $(1-8z)^{1/3}$ $\frac{1}{3}(1-8z)^{-2/3}(-8) = -\frac{8}{3}(1-8z)^{-2/3}$

4. $R(w) = \csc(7w)$ $-\csc(7w)\cot(7w)(7) = -7\csc(7w)\cot(7w)$

5. $G(x) = 2\sin(3x + \tan(x))$ $2\cos(3x+\tan(x))(3+\sec^2 x)$

6. $h(u) = \tan(4+10u)$ $10\sec^2(4+10u)$

7. $f(t) = 5 + \mathbf{e}^{4t+t^7}$ $(4+7t^6)\,e^{4t+t^7}$

8. $g(x) = \mathbf{e}^{1-\cos(x)}$ $\sin(x)e^{1-\cos(x)}$

9. $H(z) = 2^{1-6z}$ $-6\cdot 2^{1-6z}\ln(2)$

10. $u(t) = \tan^{-1}(3t-1)$ $3\cdot\frac{1}{1+(3t-1)^2} = \frac{3}{1+(3t-1)^2}$

11. $F(y) = \ln(1-5y^2+y^3)$ $\frac{1}{1-5y^2+y^3}\cdot(-10y+3y^2) = \frac{-10y+3y^2}{1-5y^2+y^3}$

12. $V(x) = \ln(\sin(x)-\cot(x))$ $\frac{1}{\sin(x)-\cot(x)}\cdot\cos(x)+\csc^2(x) = \frac{\cos(x)+\csc^2(x)}{\sin(x)-\cot(x)}$

13. $h(z) = \sin(z^6) + \sin^6(z)$ $[\sin(z)]^6$ $6z^5\cos(z^6) + 6\sin^5(z)\cos(z)$

14. $S(w) = \sqrt{7w} + \mathbf{e}^{-w}$ $\frac{1}{2}(7w)^{-1/2}\cdot 7 + e^{-w}\cdot(-1) = \frac{7}{2}(7w)^{-1/2} - e^{-w}$

15. $g(z) = 3z^7 - \sin(z^2+6)$ $21z^6 - 2z\cos(z^2+6)$

16. $f(x) = \ln(\sin(x)) - (x^4-3x)^{10}$ $\frac{1}{\sin(x)}\cdot\cos(x) - 10(x^4-3x)^9(4x^3-3) = \cot(x) - 10(x^4-3x)^9(4x^3-3)$

17. $h(t) = t^6\sqrt{5t^2-t}$ $6t^5$ $\frac{1}{2}(5t^2-t)^{-1/2}(10t-1)$ $6t^5\sqrt{5t^2-t} + \frac{1}{2}t^6(5t^2-t)^{-1/2}(10t-1)$

18. $q(t) = t^2\ln(t^5)$ $2t$ $\frac{1}{t^5}\cdot 5t^4 = \frac{5}{t}$ $2t\ln(t^5) + t^2\cdot\frac{5}{t} = 2t\ln(t^5) + 5t$

19. $g(w) = \cos(3w)\sec(1-w)$ $-3\sin(3w)$ $-1\sec(1-w)\tan(1-w)$ $-3\sin(3w)\sec(1-w) - \cos(3w)\sec(1-w)\tan(1-w)$

20. $y = \frac{\sin(3t)}{1+t^2}$ $3\cos(3t)$ $2t$ $\frac{3\cos(3t)(1+t^2) - 2t\sin(3t)}{(1+t^2)^2}$

21. $K(x) = \frac{1+\mathbf{e}^{-2x}}{x+\tan(12x)}$ $-2e^{-2x}$ $1+12\sec^2(12x)$ $\frac{-2e^{-2x}(x+\tan(12x)) - (1+e^{-2x})(1+12\sec^2(12x))}{(x+\tan(12x))^2}$

$2x\,e^x = 2xe^x + x^2e^x$

22. $f(x) = \cos\left(x^2\mathbf{e}^x\right)$ $\quad -\left(2xe^x + x^2e^x\right)\sin\left(x^2e^x\right)$

23. $z = \sqrt{5x + \tan(4x)}$ $\quad \frac{1}{2}\left(5x + \tan(4x)\right)^{-1/2}\left(5 + 4\sec^2(4x)\right)$

$5 + 4\sec^2(4x)$

24. $f(t) = \left(\mathbf{e}^{-6t} + \sin(2-t)\right)^3$ $\quad 3\left(e^{-6t} + \sin(2-t)\right)^2\left(-6e^{-6t} - \cos(2-t)\right)$

$-6e^{-6t} \quad - \cos(2-t)$

25. $g(x) = \left(\ln\left(x^2+1\right) - \tan^{-1}(6x)\right)^{10}$ $\quad 10\left(\ln(x^2+1) - \tan^{-1}(6x)\right)^9\left(\frac{2x}{x^2+1} - \frac{6}{1+36x^2}\right)$

$\frac{2x}{x^2+1} \quad \frac{6}{1+36x^2}$

inside outside

26. $h(z) = \left(\tan^4\left(z^2+1\right)\right)^4$ $\quad \frac{d}{dx}\tan(z^2+1) = 2z\sec^2(z^2+1) \cdot 4\tan^3(z^2+1) = 8z\sec^2(z^2+1)\tan^3(z^2+1)$

$\frac{d}{dx}(\sin(3x))^2 =$ outside

27. $f(x) = \left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-1}$ $\quad 2\sin(3x)\cos(3x)(3)$ $\quad -\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2}\left(4(12x)^{-2/3} + 6\sin(3x)\cos(3x)\right)$

$(12x)^{1/3}$ chain

$\frac{1}{3}(12x)^{-2/3}(12)$

see other Sheet

28. Find the tangent line to $f(x) = 4\sqrt{2x} - 6\mathbf{e}^{2-x}$ at $x = 2$.

29. Determine where $V(z) = z^4(2z-8)^3$ is increasing and decreasing.

30. The position of an object is given by $s(t) = \sin(3t) - 2t + 4$. Determine where in the interval $[0,3]$ the object is moving to the right and moving to the left.

~~31~~. Determine where $A(t) = t^2\mathbf{e}^{5-t}$ is increasing and decreasing.

~~32~~. Determine where in the interval $[-1,20]$ the function $f(x) = \ln\left(x^4 - 20x^3 - 100\right)$ is increasing and decreasing.

Implicit Differentiation

For problems 1 – 3 do each of the following.

(a) Find y' by solving the equation for y and differentiating directly.

(b) Find y' by implicit differentiation.

(c) Check that the derivatives in (a) and (b) are the same.

1. $\dfrac{x}{y^3} = 1$ $\quad xy^{-3} = 1$ $\quad y^{-3} - 3xy^{-4}y' = 0$ $\quad y' = \dfrac{y^{-3}}{3xy^{-4}} = \dfrac{y}{3x}$

product rule $\quad -3xy^4y' = -y^{-3}$

(a) $y = \sqrt[3]{x} = x^{1/3}$ $\quad$ Plug in $y = x^{1/3}$ $\quad y' = \dfrac{x^{1/3}}{3x} = \dfrac{1}{3x^{2/3}}$

$y' = \frac{1}{3}x^{-2/3}$

1

$\frac{d}{dx}\left(\frac{x}{y^3}\right)=(1)\frac{d}{dx}$ $y=\sqrt[3]{x}=x^{1/3}$

$\frac{1\cdot y^3-3xy^2y'}{y^6}=0$

$y^3-3x\,y^2y'=0$

$y'=\frac{y^3}{3xy^2}$

$y'=\frac{\cancel{(x^{1/3})^3}}{3(x^1)(x^{1/3})^2}=\frac{\cancel{x}}{3\cancel{x}(x^{2/3})}=\frac{1}{3x^{2/3}}$

$\frac{d}{dx}(xy^{-3})=(1)\frac{d}{dx}$

$y^{-3}+-3xy^{-4}y'=0$ $\rightarrow$ $\cancel{3xy^{-4}}y'=y^{-3}$

$y'=\frac{y^{-3}}{3xy^{-4}}$

$y'=\frac{(x^{1/3})^{-3}}{3x(x^{1/3})^{-4}}=\frac{x^{-1}}{3x^1(x^{-4/3})}=\frac{1}{3x^{2/3}}$

$x^{\frac{3}{3}}x^{\frac{3}{3}}x^{-4/3}$

Sample Problems

Prove each of the following identities.

1. $\tan x \sin x + \cos x = \sec x$

2. $\dfrac{1}{\tan x} + \tan x = \dfrac{1}{\sin x \cos x}$

3. $\sin x - \sin x \cos^2 x = \sin^3 x$

4. $\dfrac{\cos \alpha}{1 + \sin \alpha} + \dfrac{1 + \sin \alpha}{\cos \alpha} = 2 \sec \alpha$

5. $\dfrac{\cos x}{1 - \sin x} - \dfrac{\cos x}{1 + \sin x} = 2 \tan x$

6. $\cos^2 x = \dfrac{\csc x \cos x}{\tan x + \cot x}$

7. $\dfrac{\sin^4 x - \cos^4 x}{\sin^2 x - \cos^2 x} = 1$

8. $\dfrac{\tan^2 x}{\tan^2 x + 1} = \sin^2 x$

9. $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$

10. $1 - 2\cos^2 x = \dfrac{\tan^2 x - 1}{\tan^2 x + 1}$

11. $\tan^2 \theta = \csc^2 \theta \tan^2 \theta - 1$

12. $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$

13. $\dfrac{\csc \beta}{\sin \beta} - \dfrac{\cot \beta}{\tan \beta} = 1$

14. $\sin^4 x - \cos^4 x = 1 - 2\cos^2 x$

15. $(\sin x - \cos x)^2 + (\sin x + \cos x)^2 = 2$

16. $\dfrac{\sin^2 x + 4 \sin x + 3}{\cos^2 x} = \dfrac{3 + \sin x}{1 - \sin x}$

17. $\dfrac{\cos x}{1 - \sin x} - \tan x = \sec x$

18. $\tan^2 x + 1 + \tan x \sec x = \dfrac{1 + \sin x}{\cos^2 x}$

2. $x^2 + y^3 = 4$

$x^2 + y^3 = 4$ $\quad 3y^2y' = -2x$

$2x + 3y^2y' = 0$ $\quad y' = \frac{-2x}{3y^2}$

3. $x^2 + y^2 = 2$

$x^2 + y^2 = 2$ $\quad 2yy' = -2x$

$2x + 2yy' = 0$ $\quad y' = \frac{-x}{y}$

For problems 4 – 9 find y' by implicit differentiation. See separate sheet

4. $2y^3 + 4x^2 - y = x^6$

5. $7y^2 + \sin(3x) = 12 - y^4$

6. $\mathbf{e}^x - \sin(y) = x$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

8. $\cos(x^2 + 2y) + x\mathbf{e}^{y^2} = 1$

9. $\tan(x^2y^4) = 3x + y^2$

For problems 10 & 11 find the equation of the tangent line at the given point. See separate sheet

10. $x^4 + y^2 = 3$ at $(1, -\sqrt{2})$.

11. $y^2\mathbf{e}^{2x} = 3y + x^2$ at $(0,3)$.

For problems 12 & 13 assume that $x = x(t)$, $y = y(t)$ and $z = z(t)$ and differentiate the given equation with respect to t. leave as derivative, don't try to solve for any letter See separate sheet

12. $x^2 - y^3 + z^4 = 1$

13. $x^2\cos(y) = \sin(y^3 + 4z)$

Related Rates

1. In the following assume that x and y are both functions of t. Given $x = -2$, $y = 1$ and $x' = -4$ determine y' for the following equation.

$x=-2$
$y=1$
$x'=-4$

$y'=\frac{8}{11}$

$12yy' + 2xx' = -3x^2e^{4-4y}x' + x^3e^{4-4y}(-4)y'$

$12yy' - 4x^3e^{4-4y}y' = -3x^2e^{4-4y}x' - 2xx'$

$y' = \frac{-3x^2e^{4-4y}x' - 2xx'}{12y - 4x^3e^{4-4y}}$

$$6y^2 + x^2 = 2 - x^3\mathbf{e}^{4-4y}$$

$y' = \frac{-3(-2)^2e^{4-4(1)}(-4) - 2(-2)(-4)}{12(1) - 4(-2)^3e^{4-4(1)}} = \frac{32}{44} = \frac{8}{11}$

2. In the following assume that x, y and z are all functions of t. Given $x=4$, $y=-2$, $z=1$, $x'=9$ and $y'=-3$ determine z' for the following equation.

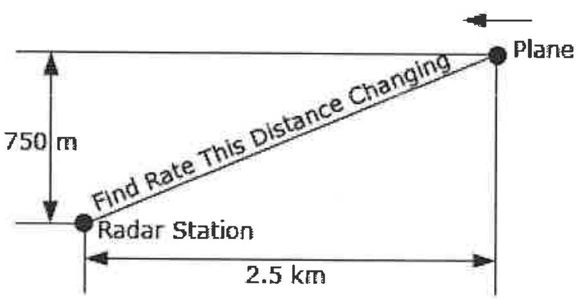
$$x(1-y)+5z^3 = y^2z^2 + x^2 - 3$$

3. For a certain rectangle the length of one side is always three times the length of the other side.
   (a) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?
   (b) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches/minute?

4. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec at what rate is the radius decreasing when the area of the sheet is 12 m^2?

5. A person is standing 350 feet away from a model rocket that is fired straight up into the air at a rate of 15 ft/sec. At what rate is the distance between the person and the rocket increasing **(a)** 20 seconds after liftoff? **(b)** 1 minute after liftoff?

6. A plane is 750 meters in the air flying parallel to the ground at a speed of 100 m/s and is initially 2.5 kilometers away from a radar station. At what rate is the distance between the plane and the radar station changing **(a)** initially and **(b)** 30 seconds after it passes over the radar station? See the (probably bad) sketch below to help visualize the problem.



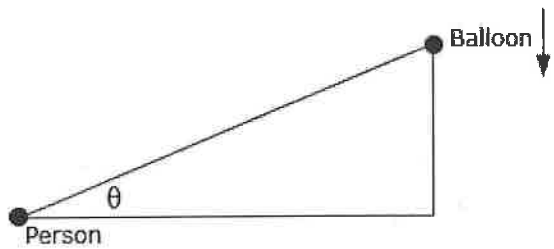
7. Two people are at an elevator. At the same time one person starts to walk away from the elevator at a rate of 2 ft/sec and the other person starts going up in the elevator at a rate of 7 ft/sec. What rate is the distance between the two people changing 15 seconds later?

8. Two people on bikes are at the same place. One of the bikers starts riding directly north at a rate of 8 m/sec. Five seconds after the first biker started riding north the second starts to ride directly east at a rate of 5 m/sec. At what rate is the distance between the two riders increasing 20 seconds after the second person started riding?

9. A light is mounted on a wall 5 meters above the ground. A 2 meter tall person is initially 10 meters from the wall and is moving towards the wall at a rate of 0.5 m/sec. After 4 seconds of moving is the tip of the shadow moving **(a)** towards or away from the person and **(b)** towards or away from the wall?

10. A tank of water in the shape of a cone is being filled with water at a rate of 12 m^3/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing with the radius of the top of the water is 10 meters?

11. The angle of elevation is the angle formed by a horizontal line and a line joining the observer's eye to an object above the horizontal line. A person is 500 feet way from the launch point of a hot air balloon. The hot air balloon is starting to come back down at a rate of 15 ft/sec. At what rate is the angle of elevation, θ, changing when the hot air balloon is 200 feet above the ground. See the (probably bad) sketch below to help visualize the angle of elevation if you are having trouble seeing it.



Higher Order Derivatives See separate Sheet

For problems 1 – 5 determine the fourth derivative of the given function.

1. $h(t)=3t^7-6t^4+8t^3-12t+18$

2. $V(x)=x^3-x^2+x-1$

3. $f(x)=4\sqrt[5]{x^3}-\frac{1}{8x^2}-\sqrt{x}$

4. $f(w)=7\sin\left(\frac{w}{3}\right)+\cos(1-2w)$

5. $y=\mathbf{e}^{-5z}+8\ln\left(2z^4\right)$

For problems 6 – 9 determine the second derivative of the given function.

6. $g(x) = \sin\left(2x^3 - 9x\right)$

7. $z = \ln\left(7 - x^3\right)$

8. $Q(v) = \dfrac{2}{\left(6 + 2v - v^2\right)^4}$

9. $H(t) = \cos^2(7t)$

For problems 10 & 11 determine the second derivative of the given function.

10. $2x^3 + y^2 = 1 - 4y$

11. $6y - xy^2 = 1$

Logarithmic Differentiation See separate sheet

For problems 1 – 3 use logarithmic differentiation to find the first derivative of the given function.

1. $f(x) = \left(5 - 3x^2\right)^7 \sqrt{6x^2 + 8x - 12}$

2. $y = \dfrac{\sin\left(3z + z^2\right)}{\left(6 - z^4\right)^3}$

3. $h(t) = \dfrac{\sqrt{5t + 8}\ \sqrt[3]{1 - 9\cos(4t)}}{\sqrt[4]{t^2 + 10t}}$

For problems 4 & 5 find the first derivative of the given function.

4. $g(w) = (3w - 7)^{4w}$

5. $f(x) = \left(2x - \mathrm{e}^{8x}\right)^{\sin(2x)}$

(27.) $\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-1}$

outside $-1\left(\sqrt[3]{12x} + \sin^2(3x)\right)^{-2}$ inside $\left(4(12x)^{-2/3} + 6\sin(3x)\cos(3x)\right)$

$(12x)^{1/3}$

$(\sin(3x))^2$

inside

$\frac{1}{3}(12x)^{-2/3}(12)$

$4(12x)^{-2/3}$

outside $2(\sin(3x))$ inside $\cos(3x)\cdot 3$

$6\sin(3x)\cos(3x)$

(28.) $f(x) = 4\sqrt{2x} - 6e^{2-x}$

$4(2x)^{1/2} - 6e^{(2-x)}\cdot(-1)$

$2(2x)^{-1/2}(2)$

$f'(x) = 4(2x)^{-1/2} + 6e^{2-x}$

$f(2) = 4\sqrt{2\cdot 2} - 6e^{2-2} = 8 - 6 = 2$

$f'(2) = 4(2\cdot 2)^{-1/2} + 6e^{2-2} = \frac{4}{2} + 6 = 8$

tan line $y = 2 + 8(x-2)$

$y = 8x - 14$

(29.) $V(z) = z^4(2z-8)^3$

$4z^3$

chain rule

outside inside

$3(2z-8)^2(2)$

$6(2z-8)^2$

Product rule $4z^3(2z-8)^3 + 6z^4(2z-8)^2$

Factor out

$2z^3(2z-8)$

$2z^3(2z-8)^2(2(2z-8) + 3z) = 0$

$2z^3(2z-8)^2(7z-16) = 0$

$z=0 \quad z=4 \quad z=16/7$

Increasing $(-\infty,0)\cup(\frac{16}{7},4)\cup(4,\infty)$

Decreasing: $(0, 16/7)$

$f'(-1)$ + | 0 | $f'(1)$ − | 16/7 | $f'(3)$ + | 4 | $f'(5)$ +

(30.) $s(t) = \sin(3t) - 2t + 4$

$s'(t) = 3\cos(3t) - 2 = 0$

$\cos(3t) = 2/3$

$\cos^{-1}(2/3) = .8411$

$3t = .8411 + 2\pi n \Rightarrow t = .2804 + 2.0944n$

$3t = 5.4421 + 2\pi n \Rightarrow t = 1.8140 + 2.0944n$

$t = 0.2804 \quad 2.3748$

1.8140

Right: $[0, 0.2804)\cup(1.8140, 2.3748)$

Left: $(0.2804, 1.8140)\cup(2.3748, 3]$

$f'(0)$ + | 0.2804 | $f'(1)$ − | 1.8140 | $f'(2)$ + | 2.3748 | $f'(3)$ −

Implicit Differentiation

(4) $2y^3 + 4x^2 - y = x^6$

$6y^2y' + 8x - y' = 6x^5$

$6y^2y' - y' = 6x^5 - 8x$

$y'(6y^2 - 1) = 6x^5 - 8x$

$y' = \dfrac{6x^5 - 8x}{6y^2 - 1}$

(5) $7y^2 + \sin(3x) = 12 - y^4$

$14yy' + 3\cos(3x) = -4y^3y'$

$3\cos(3x) = -4y^3y' - 14yy'$

$3\cos(3x) = y'(-4y^3 - 14y)$

$y' = \dfrac{3\cos(3x)}{(-4y^3 - 14y)}$

(6) $e^x - \sin(y) = x$

$e^x - \cos(y)\,y' = 1$

$-\cos(y)\,y' = 1 - e^x$

$y' = \dfrac{1 - e^x}{-\cos(y)} = -(1 - e^x)\sec(y)$

(7) $\boxed{4x^2y^7} - 2x = x^5 + 4y^3$

product

$4x^2 \cdot 7y^6y'$

$8x\,y^7 + 28x^2y^6y' - 2 = 5x^4 + 12y^2y'$

$28x^2y^6y' - 12y^2y' = 5x^4 - 8x\,y^7 + 2$

$y'(28x^2y^6 - 12y^2) = 5x^4 - 8x\,y^7 + 2$

$y' = \dfrac{5x^4 - 8x\,y^7 + 2}{28x^2y^6 - 12y^2}$

(8) $\underbrace{\cos(x^2 + 2y)}_{\text{chain}} + \underbrace{xe^{y^2}}_{\text{product \& chain}} = 1$

$-\sin(x^2 + 2y)(2x + 2y') + e^{y^2} + xe^{y^2} \cdot 2yy' = 0$

$-2x\sin(x^2 + 2y) - 2y'\sin(x^2 + 2y) + e^{y^2} + 2xyy'e^{y^2} = 0$

$-2y'\sin(x^2 + 2y) + 2xyy'e^{y^2} = 2x\sin(x^2 + 2y) - e^{y^2}$

$y'(-2\sin(x^2 + 2y) + 2xye^{y^2}) = 2x\sin(x^2 + 2y) - e^{y^2}$

$y' = \dfrac{2x\sin(x^2 + 2y) - e^{y^2}}{-2\sin(x^2 + 2y) + 2xye^{y^2}}$

(9) $\tan(x^2y^4) = 3x + y^2$

chain & product

$\sec^2(x^2y^4)(2xy^4 + 4x^2y^3y') = 3 + 2yy'$

$2xy^4\sec^2(x^2y^4) + 4x^2y^3y'\sec^2(x^2y^4) = 3 + 2yy'$

$4x^2y^3y'\sec^2(x^2y^4) - 2yy' = 3 - 2xy^4\sec^2(x^2y^4)$

$y'(4x^2y^3\sec^2(x^2y^4) - 2y) = 3 - 2xy^4\sec^2(x^2y^4)$

$y' = \dfrac{3 - 2xy^4\sec^2(x^2y^4)}{4x^2y^3\sec^2(x^2y^4) - 2y}$

(10) $x^4+y^2=3$ at $(1,-\sqrt{2})$

$4x^3+2yy'=0$

$2yy'=-4x^3$

$y'=\dfrac{-2x^3}{y}\Big|_{x=1\ y=-\sqrt{2}}$

$y'=\dfrac{-2}{-\sqrt{2}}=\sqrt{2}$

tan line: $y=-\sqrt{2}+\sqrt{2}(x-1)$

$y=\sqrt{2}x-2\sqrt{2}$

(11) $y^2e^{2x}=3y+x^2$ at $(0,3)$

product & chain

$2yy'e^{2x}+2y^2e^{2x}=3y'+2x$

$2yy'e^{2x}-3y'=2x-2y^2e^{2x}$

$y'(2ye^{2x}-3)=2x-2y^2e^{2x}$

$y'=\dfrac{2x-2y^2e^{2x}}{2ye^{2x}-3}\Big|_{x=0\ y=3}$

$y'=\dfrac{2(0)-2(3)^2e^{2(0)}}{2(3)e^{2(0)}-3}=\dfrac{-18}{3}=-6$

tan line: $y=3-6(x-0)$

$y=-6x+3$

(12) $x^2-y^3+z^4=1$

$2xx'-3y^2y'+4z^3z'=0$

(13) $x^2\cos(y)=\sin(y^3+4z)$

$2xx'\cos(y)-x^2y'\sin(y)=\cos(y^3+4z)(3y^2y'+4z')$

Higher Order Derivatives

1. $h'(t) = 21t^6 - 24t^3 + 24t^2 - 12$
$h''(t) = 126t^5 - 72t^2 + 48t$
$h'''(t) = 630t^4 - 144t$
$h^{(4)}(t) = 2520t^3 - 144$

2. $V'(x) = 3x^2 - 2x + 1$
$V''(x) = 6x - 2$
$V'''(x) = 6$
$V^{(4)}(x) = 0$

3. $f(x) = 4x^{3/5} - \frac{1}{8}x^{-2} - x^{1/2}$

$f'(x) = \frac{12}{5}x^{-2/5} + \frac{1}{4}x^{-3} - \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{-24}{25}x^{-7/5} - \frac{3}{4}x^{-4} + \frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{168}{125}x^{-12/5} + 3x^{-5} - \frac{3}{8}x^{-5/2}$

$f^{(4)}(x) = \frac{-2016}{625}x^{-17/5} - 15x^{-6} + \frac{15}{16}x^{-7/2}$

4. $f(w) = 7\sin(w/3) + \cos(1-2w)$

$f'(w) = \frac{7}{3}\cos(\frac{w}{3}) + 2\sin(1-2w)$

$f''(w) = -\frac{7}{9}\sin(\frac{w}{3}) - 4\cos(1-2w)$

$f'''(w) = \frac{-7}{27}\cos(w/3) - 8\sin(1-2w)$

$f^{(4)}(w) = \frac{7}{81}\sin(\frac{w}{3}) + 16\cos(1-2w)$

5. $y = e^{-5z} + 8\ln(2z^4)$

$y' = -5e^{-5z} + 8 \cdot \frac{8z^3}{2z^4} = -5e^{-5z} + 32z^{-1}$

$y'' = 25e^{-5z} + 32z^{-2}$

$y''' = -125e^{-5z} - 64z^{-3}$

$\frac{d^4y}{dx^4} = 625e^{-5z} - 192z^{-4}$

6. $g(x) = \sin(2x^3 - 9x)$

$g'(x) = \cos(2x^3 - 9x)(6x^2 - 9)$

product & chain

$g''(x) = -\sin(2x^3 - 9x)(6x^2 - 9)^2 +$
$12x\cos(2x^3 - 9x)$ ...

7. $z = \ln(7 - x^3)$

$\frac{dz}{dx} = \frac{-3x^2}{7 - x^3}$ $\quad \frac{-6x}{-3x^2}$

quotient rule

$\frac{d^2z}{dx^2} = \frac{-6x(7-x^3) + 3x^2(-3x^2)}{(7-x^3)^2}$

$= \frac{-42x + 6x^4 - 9x^4}{(7-x^3)^2}$

$= \frac{-42x - 3x^4}{(7-x^3)^2}$

8. $Q(v) = \frac{2}{(6+2v-v^2)^4} = 2(6+2v-v^2)^{-4}$

$Q'(v) = -8(6+2v-v^2)^{-5}(2-2v)$

product rule

$Q''(v) = 40(6+2v-v^2)^{-6}(2-2v)^2 +$
$16(6+2v-v^2)^{-5}$

9. $H'(t) = 2\cos(7t) \cdot -\sin(7t) \cdot 7$
$= -14\sin(7t)\cos(7t)$

product rule

$H''(t) = -14\cos(7t) \cdot 7 \cdot \cos(7t) +$
$-14\sin(7t) \cdot -\sin(7t) \cdot 7$

$= -98\cos^2(7t) + 98\sin^2(7t)$

10. $2x^3+y^2=1-4y$

$\frac{dy}{dx}$ $6x^2+2yy'=-4y'$

$2yy'+4y'=-6x^2$

$y'(2y+4)=-6x^2$

$y'=\frac{-6x^2}{2y+4}=\frac{-3x^2}{y+2}=-3x^2(y+2)^{-1}$

for y'':

$y''=-6x(y+2)^{-1}+(-3x^2)(-1)(y+2)^{-2}\cdot \underset{\uparrow \text{plug in}}{y'}$

$=-6x(y+2)^{-1}+3x^2(y+2)^{-2}\cdot\frac{-3x^2}{(y+2)}$

$=-6x(y+2)^{-1}-9x^4(y+2)^{-3}$

11. $\frac{dy}{dx}\left[\boxed{6y-xy^2=1}\right]$

$6y'-y^2-2xyy'=0$

$6y'-2xyy'=+y^2$

$y'(6-2xy)=+y^2$

$y'=\frac{-y^2}{6-2xy}=+y^2(6-2xy)^{-1}$

product rule

$y''=2yy'(6-2xy)^{-1}+y^2(-1)(6-2xy)^{-2}(-2xy'-2y)$

$=2y\cdot y^2(6-2xy)^{-1}(6-2xy)^{-1}-y^2(6-2xy)^{-2}(-2x\cdot y^2\cdot(6-2xy)^{-1}-2y)$

$=2y^3(6-2xy)^{-2}-y^2(6-2xy)^{-2}(-2xy^2(6-2xy)^{-1}-2y)$

Logarithmic Differentiation

1. $f(x) = (5-3x^2)^7\sqrt{6x^2+8x-12}$

$\ln[f(x)] = \ln\left[(5-3x^2)^7\sqrt{6x^2+8x-12}\right]$

$\ln[f(x)] = \ln(5-3x^2)^7 + \ln(6x^2+8x-12)^{1/2}$

$\frac{dy}{dx}\left[\ln[f(x)] = 7\ln(5-3x^2) + \frac{1}{2}\ln(6x^2+8x-12)\right]\frac{dy}{dx}$

$f'(x)\cdot\frac{1}{f(x)} = 7\cdot\frac{-6x}{5-3x^2} + \frac{1}{2}\cdot\frac{\overset{6}{\cancel{12}}x+\overset{4}{\cancel{8}}}{6x^2+8x-12}$

$f'(x) = f(x)\cdot\ \frac{-42x}{5-3x^2} + \frac{3x+2}{3x^2+4x-6}$

$\uparrow$ plugin

$f'(x) = (5-3x^2)^7\sqrt{6x^2+8x-12}\left(\frac{-42x}{5-3x^2} + \frac{3x+2}{3x^2+4x-6}\right)$

2. $y = \frac{\sin(3z+z^2)}{(6-z^4)^3}$

$\frac{dy}{dx}\left[\ln y\right] = \left[\ln\sin(3z+z^2) - 3\ln(6-z^4)\right]\frac{dy}{dx}$

$\frac{y'}{y} = \frac{(3+2z)\cos(3z+z^2)}{\sin(3z+z^2)} - \frac{3(-4z^3)}{6-z^4}$

$y' = \left(\frac{\sin(3z+z^2)}{(6-z^4)^3}\right)\cdot\left((3+2z)\cot(3z+z^2) + \frac{12z^3}{6-z^4}\right)$

3. $\ln(h(t))=\ln\left(\frac{\sqrt{5t+8}\ \sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}\right)$

$\frac{dy}{dx}\left[\ln h(t)\right] = \left[\frac{1}{2}\ln(5t+8) + \frac{1}{3}\ln(1-9\cos(4t)) - \frac{1}{4}\ln(t^2+10t)\right]\frac{dy}{dx}$

$\frac{h'(t)}{h(t)} = \frac{1}{2}\cdot\frac{5}{5t+8} + \frac{1}{3}\cdot\frac{36\sin(4t)}{1-9\cos(4t)} - \frac{1}{4}\cdot\frac{t+10}{t^2+10t}$

$h'(t) = \left(\frac{\sqrt{5t+8}\ \sqrt[3]{1-9\cos(4t)}}{\sqrt[4]{t^2+10t}}\right)\cdot\left(\frac{5}{2(5t+8)} + \frac{12\sin(4t)}{(1-9\cos(4t))} - \frac{t+10}{4(t^2+10t)}\right)$

$4.\ \ln\left(g(w)\right) = \ln\left((3w-7)^{4w}\right)$

$\frac{dy}{dx}\left[\ln g(w)\right] = \left[4w \ln(3w-7)\right]\frac{dy}{dx}$

product rule

$$\frac{g'(w)}{g(w)} = 4\ln(3w-7) + 4w \cdot \frac{3}{3w-7}$$

$$g'(w) = (3w-7)^{4w}\left(4\ln(3w-7) + \frac{12w}{3w-7}\right)$$

$5.\ \ln\left(f(x)\right) = \ln\left((2x-e^{8x})^{\sin(2x)}\right)$

$\frac{dy}{dx}\left[\ln f(x)\right] = \left[\sin(2x)\ln(2x-e^{8x})\right]\frac{dy}{dx}$

product rule

$$\frac{f'(x)}{f(x)} = 2\cos(2x)\ln(2x-e^{8x}) + \sin(2x) \cdot \frac{2-8e^{8x}}{2x-e^{8x}}$$

$$f'(x) = (2x-e^{8x})^{\sin(2x)}\left(2\cos(2x)\ln(2x-e^{8x}) + \frac{\sin(2x)(2-8e^{8x})}{2x-e^{8x}}\right)$$