Related Rates

From derivatives Unit packet

2. $x^2 + y^3 = 4$ $x^2 + y^3 = 4$ $2x + 3y^2y' = 0$ $3y^2y' = -2x$ $y' = \frac{-2x}{3y^2}$

3. $x^2 + y^2 = 2$ $x^2 + y^2 = 2$ $2x + 2yy' = 0$ $2yy' = -2x$ $y' = \frac{-x}{y}$

For problems 4 – 9 find y' by implicit differentiation. See separate sheet

4. $2y^3 + 4x^2 - y = x^6$

5. $7y^2 + \sin(3x) = 12 - y^4$

6. $\mathbf{e}^x - \sin(y) = x$

7. $4x^2y^7 - 2x = x^5 + 4y^3$

8. $\cos(x^2 + 2y) + x\,\mathbf{e}^{y^2} = 1$

9. $\tan(x^2y^4) = 3x + y^2$

For problems 10 & 11 find the equation of the tangent line at the given point. See separate sheet

10. $x^4 + y^2 = 3$ at $(1, -\sqrt{2})$.

11. $y^2\mathbf{e}^{2x} = 3y + x^2$ at $(0, 3)$.

For problems 12 & 13 assume that $x = x(t)$, $y = y(t)$ and $z = z(t)$ and differentiate the given equation with respect to t. leave as derivative, don't try to solve for any letter See separate sheet

12. $x^2 - y^3 + z^4 = 1$

13. $x^2\cos(y) = \sin(y^3 + 4z)$

Related Rates

1. In the following assume that x and y are both functions of t. Given $x = -2$, $y = 1$ and $x' = -4$ determine y' for the following equation.

$x=-2$
$y=1$
$x'=-4$

$\boxed{y'=\frac{8}{11}}$

$$6y^2+x^2=2-x^3\mathbf{e}^{4-4y}$$

$12yy'+2xx'=-3x^2e^{4-4y}x'++x^3e^{4-4y}(-4)y'$

$12yy'-4x^3e^{4-4y}y'=-3x^2e^{4-4y}x'-2xx'$

$y'=\frac{-3x^2e^{4-4y}x'-2xx'}{12y-4x^3e^{4-4y}}$

$y'=\frac{-3(-2)^2e^{4-4(1)}(-4)-2(-2)(-4)}{12(1)-4(-2)^3e^{4-4(1)}}=\frac{32}{44}=\frac{8}{11}$

2. In the following assume that *x*, *y* and *z* are all functions of *t*. Given $x=4$, $y=-2$, $z=1$, $x'=9$ and $y'=-3$ determine z' for the following equation.

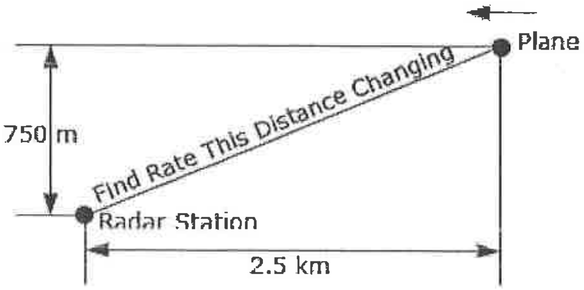
$$x(1-y)+5z^3=y^2z^2+x^2-3$$

3. For a certain rectangle the length of one side is always three times the length of the other side.
   (a) If the shorter side is decreasing at a rate of 2 inches/minute at what rate is the longer side decreasing?
   (b) At what rate is the enclosed area decreasing when the shorter side is 6 inches long and is decreasing at a rate of 2 inches/minute?

4. A thin sheet of ice is in the form of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.5 m^2/sec at what rate is the radius decreasing when the area of the sheet is 12 m^2?

5. A person is standing 350 feet away from a model rocket that is fired straight up into the air at a rate of 15 ft/sec. At what rate is the distance between the person and the rocket increasing **(a)** 20 seconds after liftoff? **(b)** 1 minute after liftoff?

6. A plane is 750 meters in the air flying parallel to the ground at a speed of 100 m/s and is initially 2.5 kilometers away from a radar station. At what rate is the distance between the plane and the radar station changing **(a)** initially and **(b)** 30 seconds after it passes over the radar station? See the (probably bad) sketch below to help visualize the problem.



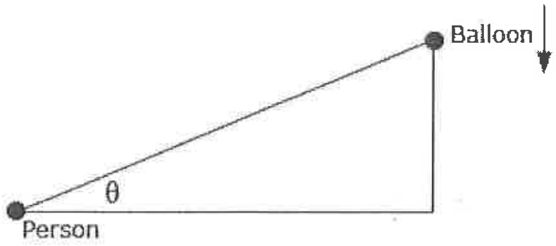
7. Two people are at an elevator. At the same time one person starts to walk away from the elevator at a rate of 2 ft/sec and the other person starts going up in the elevator at a rate of 7 ft/sec. What rate is the distance between the two people changing 15 seconds later?

8. Two people on bikes are at the same place. One of the bikers starts riding directly north at a rate of 8 m/sec. Five seconds after the first biker started riding north the second starts to ride directly east at a rate of 5 m/sec. At what rate is the distance between the two riders increasing 20 seconds after the second person started riding?

9. A light is mounted on a wall 5 meters above the ground. A 2 meter tall person is initially 10 meters from the wall and is moving towards the wall at a rate of 0.5 m/sec. After 4 seconds of moving is the tip of the shadow moving **(a)** towards or away from the person and **(b)** towards or away from the wall?

10. A tank of water in the shape of a cone is being filled with water at a rate of 12 m^3/sec. The base radius of the tank is 26 meters and the height of the tank is 8 meters. At what rate is the depth of the water in the tank changing with the radius of the top of the water is 10 meters?

11. The angle of elevation is the angle formed by a horizontal line and a line joining the observer's eye to an object above the horizontal line. A person is 500 feet way from the launch point of a hot air balloon. The hot air balloon is starting to come back down at a rate of 15 ft/sec. At what rate is the angle of elevation, θ, changing when the hot air balloon is 200 feet above the ground. See the (probably bad) sketch below to help visualize the angle of elevation if you are having trouble seeing it.



Higher Order Derivatives

See separate sheet

For problems 1 – 5 determine the fourth derivative of the given function.

1. $h(t) = 3t^7 - 6t^4 + 8t^3 - 12t + 18$

2. $V(x) = x^3 - x^2 + x - 1$

3. $f(x) = 4\sqrt[5]{x^3} - \frac{1}{8x^2} - \sqrt{x}$

4. $f(w) = 7\sin\left(\frac{w}{3}\right) + \cos(1 - 2w)$

5. $y = \mathbf{e}^{-5z} + 8\ln\left(2z^4\right)$

For problems 6 – 9 determine the second derivative of the given function.

6. $g\left(x\right)=\sin\left(2x^3-9x\right)$

7. $z=\ln\left(7-x^3\right)$

8. $Q\left(v\right)=\dfrac{2}{\left(6+2v-v^2\right)^4}$

9. $H\left(t\right)=\cos^2\left(7t\right)$

For problems 10 & 11 determine the second derivative of the given function.

10. $2x^3+y^2=1-4y$

11. $6y-xy^2=1$

Logarithmic Differentiation

See separate sheet

For problems 1 – 3 use logarithmic differentiation to find the first derivative of the given function.

1. $f\left(x\right)=\left(5-3x^2\right)^7\ \sqrt{6x^2+8x-12}$

2. $y=\dfrac{\sin\left(3z+z^2\right)}{\left(6-z^4\right)^3}$

3. $h\left(t\right)=\dfrac{\sqrt{5t+8}\ \sqrt[3]{1-9\cos\left(4t\right)}}{\sqrt[4]{t^2+10t}}$

For problems 4 & 5 find the first derivative of the given function.

4. $g\left(w\right)=\left(3w-7\right)^{4w}$

5. $f\left(x\right)=\left(2x-\mathbf{e}^{8x}\right)^{\sin\left(2x\right)}$

Related Rates

② $x(1-y)+5z^3=y^2z^2+x^2-3$ $\quad x=4\ y=-2\ z=1\ x'=9\ y'=-3$

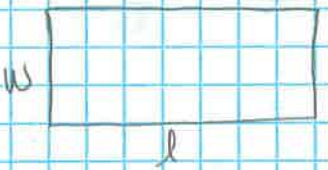
$1\cdot(1-y)x'+x(-1)y'+\underline{\underline{15z^2z'}}=2yz^2y'+\underline{\underline{y^2(2z)z'}}+2xx'$

$15z^2z'-2y^2zz'=2yz^2y'+2xx'-(1-y)x'+xy'$

$$z'=\frac{2yz^2y'+2xx'-(1-y)x'+xy'}{15z^2-2y^2z}=\frac{2(-2)(1)^2(-3)+2(4)(9)-(1+2)(9)+(4)(-3)}{15(1)^2-2(-2)^2(1)}$$

$\boxed{z'=\frac{45}{7}}$

③



ℓ = longer side
w = shorter side

$\ell=3w$

$\frac{d\ell}{dt}=3\frac{dw}{dt}$

a) $\frac{d\ell}{dt}=3(-2)=-6$

The longer side decreases at 6 in/min.

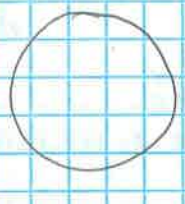
$A=\ell w$

$A=(3w)w=3w^2$

b) $\frac{dA}{dt}=6w\frac{dw}{dt}=6(6)(-2)$

The area is decreasing at 72 in²/min

④



$\frac{dA}{dt}=-0.5\ m^2/sec$ $\quad \frac{dr}{dt}=?$ $\quad A=12m^2\rightarrow A=\pi r^2$

$12=\pi r^2$

$12/\pi=r^2$

$2\sqrt{3}\pi^{-1/2}=2\sqrt{3}/\sqrt{\pi}=r$

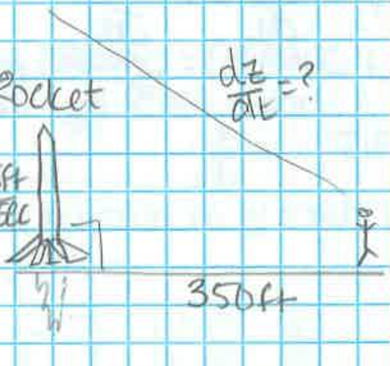
$A=\pi r^2$

$\frac{dA}{dt}=2\pi r\frac{dr}{dt}\Rightarrow -0.5=2\pi\cdot 2\sqrt{3}\pi^{-1/2}\frac{dr}{dt}\Rightarrow -0.5=4\sqrt{3}\pi^{1/2}\frac{dr}{dt}$

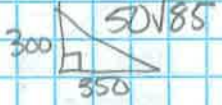
$\frac{dr}{dt}=-0.0407168876$

The radius is decreasing at .0407168876 m/sec

⑤ Rocket



a) 20 sec = 300 ft = R



$\frac{dz}{dt}=?$

$R^2+b^2=z^2$

$2R\frac{dR}{dt}+2b\frac{db}{dt}^{0}=2z\frac{dz}{dt}$

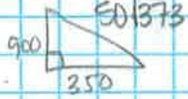
$2(300)(15)=2(50\sqrt{85})\frac{dz}{dt}$

$9000=100\sqrt{85}\frac{dz}{dt}$

$\frac{90}{\sqrt{85}}=\boxed{\frac{18\sqrt{85}}{17}=\frac{dz}{dt}}$

a) The distance between the person & the rocket is increasing at 9.761870601 ft/sec

b) 60 sec = 900 ft = R



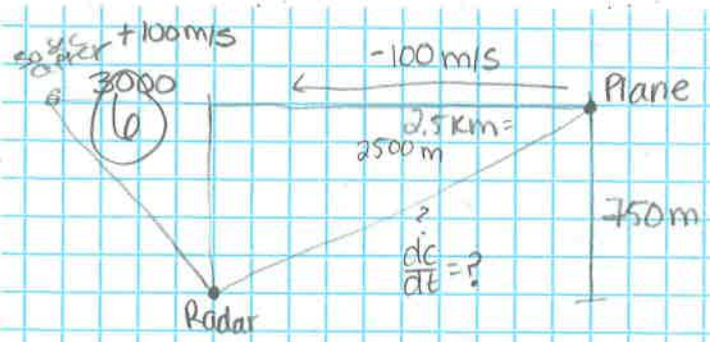
$2(900)(15)=2(50\sqrt{373})\frac{dz}{dt}$

$27000=100\sqrt{373}\frac{dz}{dt}$

$\frac{270\sqrt{373}}{373}=\frac{dz}{dt}$

b) The distance between the person & the rocket is increasing at 13.98007007 ft/sec

⑥



a) $2500^2 + 750^2 = c^2$ $\quad a^2+b^2=c^2$

$6812500 = c^2$ $\quad 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

$2610.08 = c$

$2(750)(0) + 2(2500)(-100)$

$0 - 500000 = 5220.16\frac{dc}{dt} = 2(2610.08)\frac{dc}{dt}$

$-95.7825 = \frac{dc}{dt}$

The distance between the plane and the radar station is decreasing at a rate of 95.7825 m/sec.

b) $750^2 + 3000^2 = c^2$

$9562500 = c^2$

$3092.329219 = c$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

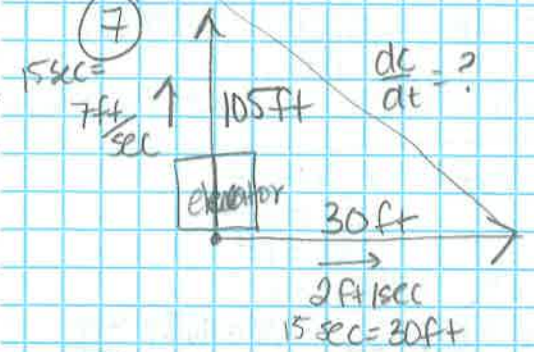
$2(3000)(100) + 2(750)(0) = 2(3092.329219)\frac{dc}{dt}$

$600000 = 2(3092.329219)\frac{dc}{dt}$

$97.01425 = \frac{dc}{dt}$

The distance between the plane and radar station 30 sec. after it passes is increasing at a rate of 97.01425 m/s

⑦



$30^2 + 105^2 = c^2$

$11925 = c^2$

$109.2016 = c$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

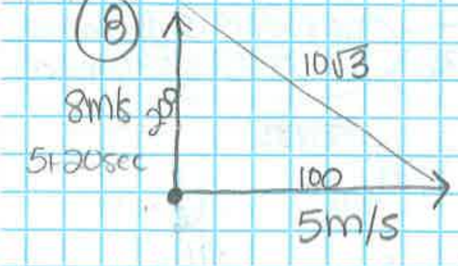
$2(105)(7) + 2(30)(2) = 2(109.2016)\frac{dc}{dt}$

$1470 + 120 = 218.4032\frac{dc}{dt}$

$7.28 = \frac{dc}{dt}$

The distance between the people is increasing at a rate of 7.28 ft/sec

⑧



$200^2 + 100^2 = c^2$

$100\sqrt{5} = c$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

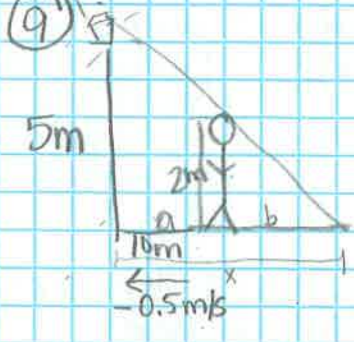
$2(200)(8) + 2(100)(5) = 2(100\sqrt{5})\frac{dc}{dt}$

$3200 + 1000 = 200\sqrt{5}\frac{dc}{dt}$

$9.3915 = \frac{dc}{dt}$

The distance between the riders is increasing at a rate of 9.3915 m/s.

⑨



$\frac{2}{b} = \frac{5}{a+b}$ $\quad x = a + b$ $\quad \frac{dx}{dt} = \frac{5}{3}\frac{da}{dt} = \frac{5}{3}\cdot\left(\frac{-1}{2}\right) = \frac{-5}{6}$

$x = a + \frac{2a}{3}$

$5b = 2a + 2b$ $\quad x = \frac{5a}{3}$

$3b = 2a$

$b = \frac{2a}{3}$

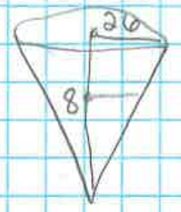
$\frac{db}{dt} = \frac{2}{3}\frac{da}{dt}$

$= \frac{2}{3}\left(\frac{-1}{2}\right) = \frac{-1}{3}$

The tip of the shadow is moving toward the wall and toward the person

Related Rates cont'd

⑩



$\frac{dV}{dt} = +12\frac{m^3}{sec}$ $\frac{dh}{dt} = ?$ $r=10$

$\frac{26}{8} = \frac{10}{h} \Rightarrow h = 3.0769$

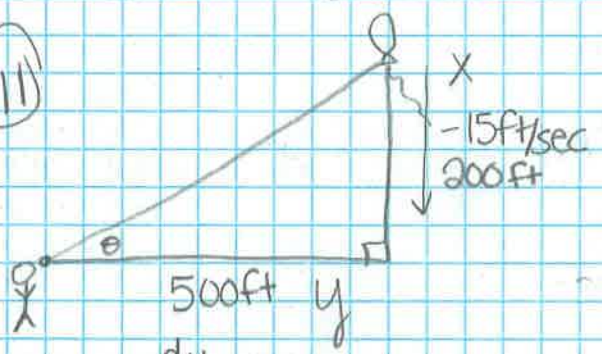
$\frac{26}{8} = \frac{r}{h}$ so, $r = \frac{26h}{8}$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{26h}{8}\right)^2 h = \frac{169}{48}\pi h^3$$

$$\frac{dV}{dt} = \frac{169}{16}\pi h^2 \frac{dh}{dt} \Rightarrow 12 = \frac{169}{16}\pi(3.0769)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = .0381978 \text{ m/s}$$

The height of the water in the tank is increasing at a rate of 0.0381978 m/s when the radius of the tank is 10 m.

⑪



$\frac{dy}{dt} = 0$

$\frac{d\theta}{dt} = ?$

$\tan\theta = \frac{200}{500}$

$\theta = .380506$

$$\tan\theta = \frac{x}{y} = xy^{-1}$$

$$\sec^2\theta \frac{d\theta}{dt} = y^{-1}\frac{dx}{dt} \cancel{-xy^{-2}\overset{0}{\frac{dy}{dt}}}$$

$$\sec^2(.380506)\frac{d\theta}{dt} = \frac{1}{500}(-15)$$

$$1.159999\frac{d\theta}{dt} = \frac{-3}{100}$$

$$\frac{d\theta}{dt} = -0.02586$$

The angle of elevation of the balloon is decreasing at a rate of 0.02586 ft/sec.

53-56, 58-62,
64-66
Business Applications
68

Applications of Derivatives

Introduction

Here are a set of practice problems for the Applications of Derivatives chapter of my Calculus I notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a number of places on the site.

13. If you'd like a pdf document containing the solutions go to the note page for the section you'd like solutions for and select the download solutions link from there. Or,

14. Go to the download page for the site http://tutorial.math.lamar.edu/download.aspx and select the section you'd like solutions for and a link will be provided there.

15. If you'd like to view the solutions on the web or solutions to an individual problem you can go to the problem set web page, select the problem you want the solution for. At this point I do not provide pdf versions of individual solutions, but for a particular problem you can select "Printable View" from the "Solution Pane Options" to get a printable version.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have practice problems written for them.

Rates of Change

As noted in the text for this section the purpose of this section is only to remind you of certain types of applications that were discussed in the previous chapter. As such there aren't any problems written for this section. Instead here is a list of links (note that these will only be active links in the web version and not the pdf version) to problems from the relevant sections from the previous chapter.

Each of the following sections has a selection of increasing/decreasing problems towards the bottom of the problem set.

Differentiation Formulas
Product & Quotient Rules
Derivatives of Trig Functions
Derivatives of Exponential and Logarithm Functions
Chain Rule

Related Rates problems are in the **Related Rates** section.

Critical Points

Determine the critical points of each of the following functions.

F 1. $f(x)=8x^3+81x^2-42x-8$ $f'(x)=24x^2+162x-42$ $f'(x)=6(x+7)(4x-1)=0$ $\boxed{x=-7,\ x=\frac{1}{4}}$

F 2. $R(t)=1+80t^3+5t^4-2t^5$ $R'(t)=-10t^4+20t^3+240t^2$ $R'(t)=-10t^2(t+4)(t-6)$ $\boxed{t=-4,\ t=6,\ t=0}$

Q 3. $g(w)=2w^3-7w^2-3w-2$ $g'(w)=6w^2-14w-3$ $\boxed{w=\frac{14\pm\sqrt{268}}{12}=\frac{7\pm\sqrt{67}}{6}}$

F, Q 4. $g(x)=x^6-2x^5+8x^4$ $g'(x)=6x^5-10x^4+32x^3$ $g'(x)=2x^3(3x^2-5x+16)$ $\cancel{x=\frac{5\pm i\sqrt{167}}{6}}$ complex, $\boxed{x=0}$

5. $h(z)=4z^3-3z^2+9z+12$ $h'(z)=12z^2-6z+9$ $h'(z)=3(4z^2-2z+3)$ $h=\cancel{\frac{1\pm i\sqrt{11}}{4}}$ $\boxed{\text{no critical points}}$

C, P F, Q 6. $Q(x)=(2-8x)^4(x^2-9)^3$ $Q'(x)=4(2-8x)^3\cdot-8\cdot(x^2-9)^3+(2-8x)^4\cdot3(x^2-9)^2\cdot2x$ $\boxed{x=\frac{1}{4},\ \pm3,\ \frac{3\pm\sqrt{5769}}{40}}$

$Q'(x)=2(2-8x)^3(x^2-9)^2[-16(x^2-9)+3x(2-8x)]$

$Q'(x)=2(2-8x)^3(x^2-9)^2(-40x^2+6x+144)=-4(2-8x)^3(x^2-9)^2(20x^2-3x-72)$

gcf 2

Q Q 7. $f(z)=\dfrac{z+4}{\underset{4z+1}{2z^2+z+8}}$ $f'(z)=\dfrac{(1)(2z^2+z+8)-(z+4)(4z+1)}{(2z^2+z+8)^2}=\dfrac{-2z^2-16z+4}{(2z^2+z+8)^2}=\dfrac{-2(z^2+8z-2)}{(2z^2+z+8)^2}$ $\boxed{x=-4\pm3\sqrt{2}}$ $\cancel{x=\frac{-1\pm i\sqrt{63}}{4}}$

Q F, Q 8. $R(x)=\dfrac{\overset{-1}{1-x}}{\underset{2x+2}{x^2+2x-15}}$ $R'(x)=\dfrac{-1(x^2+2x-15)-(1-x)(2x+2)}{(x^2+2x-15)^2}=\dfrac{x^2-2x+13}{(x^2+2x-15)^2}$ $\longrightarrow x=1\pm2i\sqrt{3}$ $\boxed{\text{no critical points}}$

$\longrightarrow (x-3)(x+5)\Rightarrow x\neq-5,\ x\neq3$ Function DNE

C F 9. $r(y)=\sqrt[5]{y^2-6y}$ $(y^2-6y)^{1/5}$

$r'(y)=\frac{1}{5}(y^2-6y)^{-4/5}(2y-6)$ = $y^2-6y=0$ $y(y-6)=0$ $y=0$ $y=6$ $2y-6=0$ $y=3$ $\boxed{y=6,\ y=3,\ y=0}$

P, C 10. $h(t)=\overset{0}{15}-(\underset{1}{3-t})\left[\underset{2t-8}{t^2-8t+7}\right]^{\frac{1}{3}}$ $(-3+t)$

$h'(t)=1(t^2-8t+7)^{1/3}+(-3+t)\cdot\frac{1}{3}(t^2-8t+7)^{-2/3}(2t-8)$

$=(t^2-8t+7)^{1/3}+\frac{(t-3)(2t-8)}{3(t^2-8t+7)^{2/3}}=\frac{3(t^2-8t+7)+(t-3)(2t-8)}{3(t^2-8t+7)^{2/3}}=\frac{5t^2-38t+45}{3(t^2-8t+7)^{2/3}}$ Q F

common denominator

$(t-7)(t-1)$

$\boxed{t=\frac{19\pm2\sqrt{34}}{5},\ t=1,\ t=7}$

11. $s(z)=4\cos(z)-z$ $S'(z)=-4\sin(z)-1$

$\sin(z)=-\frac{1}{4}$ $\sin^{-1}(-\frac{1}{4})=-0.2527$ use pos value $2\pi-0.257=6.0305$

2nd point!!! $\sin=\pi-\theta$ $\pi+0.2527$ 3.3942

$\boxed{\begin{array}{l}z=3.3942+2\pi n\\ z=6.0305+2\pi n\end{array}}$

12. $f(y)=\sin\left(\frac{y}{3}\right)+\frac{2y}{9}$ $f'(y)=\frac{1}{3}\cos\left(\frac{y}{3}\right)+\frac{2}{9}$

$\cos\left(\frac{y}{3}\right)=-\frac{2}{3}$ $\cos^{-1}(-\frac{2}{3})=2.3005$ $2\pi-2.3005=3.9827$

$\frac{y}{3}=2.3005+2\pi n$ $\frac{y}{3}=3.9827+2\pi n$ $\Rightarrow$ $\boxed{\begin{array}{l}y=6.9016+6\pi n\\ y=11.9481+6\pi n\end{array}}$

C 13. $V(t)=(\sin^2(3t))^2+1$ $V'(t)=2\sin(3t)\cos(3t)\cdot3$

$=6\sin(3t)\cos(3t)$

$\sin(3t)=0$ $3t=0+2\pi n$ $3t=\pi+2\pi n$

$\cos(3t)=0$ $3t=\pi/2+2\pi n$ $3t=\frac{3\pi}{2}+2\pi n$ $\Rightarrow$

$\boxed{\begin{array}{l}t=0+\frac{2}{3}\pi n,\ t=\frac{\pi}{3}+\frac{2}{3}\pi n\\ t=\frac{\pi}{6}+\frac{2\pi n}{3},\ t=\frac{\pi}{2}+\frac{2}{3}\pi n\end{array}}$

P, C 14. $f(x)=\underbrace{5x}\underbrace{\mathbf{e}^{9-2x}}$ $f'(x)=5e^{9-2x}+5xe^{9-2x}(-2)$

$f'(x)=5e^{9-2x}(1-2x)$ $e^{9-2x}\neq0$ $1-2x=0$ $\boxed{x=\frac{1}{2}}$

C F 15. $g(w)=\mathbf{e}^{w^3-2w^2-7w}$ $g'(w)=e^{w^3-2w^2-7w}(3w^2-4w-7)$

$e^{\text{power}}\neq0$ $3w^2-4w-7=0$ $(3w-7)(w+1)=0$ $\boxed{w=\frac{7}{3},\ w=-1}$

C 16. $R(x)=\ln\left(x^2+4x+14\right)$ $R'(x)=\frac{2x+4}{x^2+4x+14}$ $2x+4=0$ $\boxed{x=-2}$

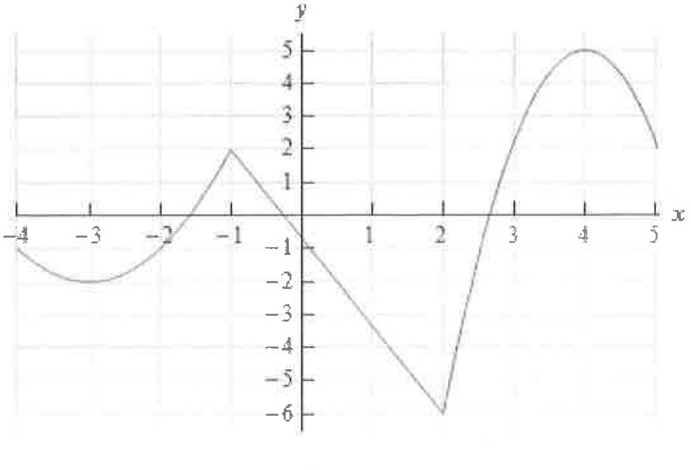
If this was 0, the argument of ln = 0 (not part of ln's domain)

17. $A(t)=3t-7\ln(8t+2)$ $A'(t)=3-\frac{7(8)}{8t+2}=3-\frac{56}{8t+2}=\frac{3(8t+2)-56}{8t+2}=\frac{24t-50}{8t+2}$ $\boxed{t=\frac{25}{12}}$ ~~$t=-\frac{1}{4}$~~

Causes ln argument to = 0

Minimum and Maximum Values

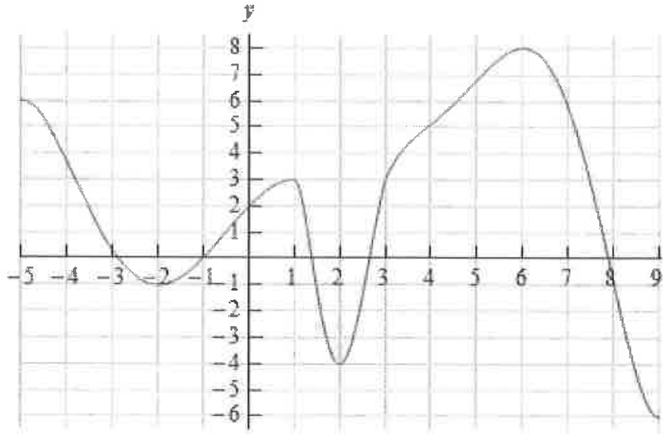
1. Below is the graph of some function, $f(x)$. Identify all of the relative extrema and absolute extrema of the function.



Absolute max (4,5)
Absolute min (2,−6)
Relative max (−1,2) and (4,5)
Relative min (−3,−2) and (2,−6)

2. Below is the graph of some function, $f(x)$. Identify all of the relative extrema and absolute extrema of the function.

Relative extrema do <u>not</u> occur at endpoints



Absolute max: (6,8)
Absolute Min: (9,-6)
Relative max:
(1, 3)
(6,8)
*Relative Min: (-2,-1)
(2,-4)

3. Sketch the graph of $g(x)=x^2-4x$ and identify all the relative extrema and absolute extrema of the function on each of the following intervals. Graph with calculator

(a) $(-\infty,\infty)$ no rel. or abs. max, rel min (2,-4), abs min (2,-4)

(b) $[-1,4]$ abs max: (-1,5) rel min / abs min (2,-4) no rel max

(c) $[1,3]$ abs max (1,-3) <u>and</u> (3,-3) rel min / abs min (2,-4) no rel max

(d) $[3,5]$ abs max (5,5) abs min (3,-3) no rel max or min

(e) $(-1,5]$ rel min / abs min (2,-4) no rel max abs max (5,5)

4. Sketch the graph of $h(x)=-(x+4)^3$ and identify all the relative extrema and absolute extrema of the function on each of the following intervals.

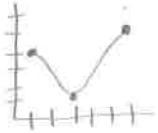
(a) $(-\infty,\infty)$ no abs min or max, no rel max or min

(b) $[-5.5,-2]$ no rel min or max, abs max (-5.5,3.375), abs min (-2,-8)

(c) $[-4,-3)$ no rel min or max, abs max (-4,0), no abs min

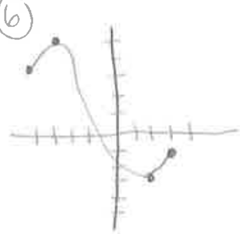
(d) $[-4,-3]$ no rel min or max, abs max (-4,0), abs min (-3,-1)

⑤



5. Sketch the graph of some function on the interval $[1,6]$ that has an absolute maximum at $x=6$ and an absolute minimum at $x=3$.

⑥



6. Sketch the graph of some function on the interval $[-4,3]$ that has an absolute maximum at $x=-3$ and an absolute minimum at $x=2$.

7. Sketch the graph of some function that meets the following conditions :

(a) The function is continuous.

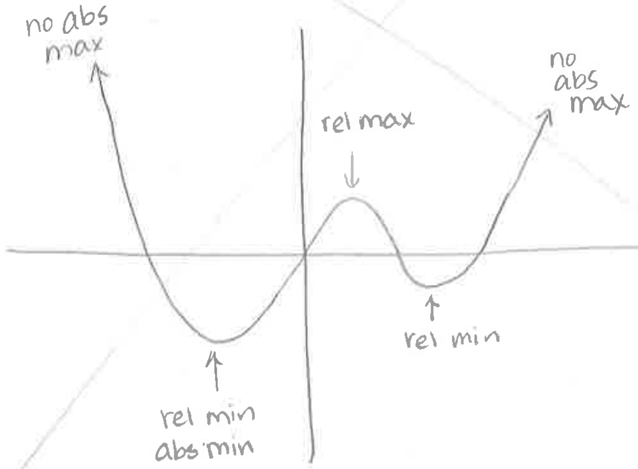
(b) Has two relative minimums.

(b) One of relative minimums is also an absolute minimum and the other relative minimum is not an absolute minimum.

(c) Has one relative maximum.

(d) Has no absolute maximum.

⑦



2 rel min
↳ one absolute
1 rel max
↳ no absolute

EVT

Finding Absolute Extrema See critical points section!

For each of the following problems determine the absolute extrema of the given function on the specified interval.

#1 1. $f(x)=8x^3+81x^2-42x-8$ on $[-8,2]$ Critical in interval $x=-7, x=\frac{1}{4}$

$f(-8)=1416$ $f(-7)=1511$ $f(\frac{1}{4})=-13.3125$ $f(2)=296$

abs max: $(-7,1511)$ abs min: $(\frac{1}{4},-13.3125)$

#1 2. $f(x)=8x^3+81x^2-42x-8$ on $[-4,2]$ $x=\frac{1}{4}$

$f(-4)=944$ $f(\frac{1}{4})=-13.3125$ $f(2)=296$

abs max: $(-4,944)$ abs min: $(\frac{1}{4},-13.3125)$

#2 3. $R(t)=1+80t^3+5t^4-2t^5$ on $[-4.5,4]$ $t=-4$ $t=0$

$R(-4.5)=-1548.13$ $R(-4)=-1791$ $R(0)=1$ $R(4)=4353$

abs max: $(4,4353)$ abs min: $(-4,-1791)$

#2 4. $R(t)=1+80t^3+5t^4-2t^5$ on $[0,7]$ $t=0$ $t=6$

$R(0)=1$ $R(6)=8209$ $R(7)=5832$

abs max: $(6,8209)$ abs min: $(0,1)$

#5 5. $h(z)=4z^3-3z^2+9z+12$ on $[-2,1]$ no critical points

$h(-2)=-50$ $h(1)=22$

abs max: $(1,22)$ abs min $(-2,-50)$

6. $g(x)=3x^4-26x^3+60x^2-11$ on $[1,5]$ See separate sheet

#6 7. $Q(x)=(2-8x)^4(x^2-9)^3$ on $[-3,3]$ $x=\frac{1}{4}, x=3$, $x=-3$, $x=\frac{3+\sqrt{5769}}{40}\approx 1.9739$, $x=\frac{3-\sqrt{5769}}{40}\approx -1.8239$

$Q(-3)=0$ $Q(-1.8239)=-1.38\times10^7$

$Q(\frac{1}{4})=0$ $Q(1.9739)=-4.81\times10^6$

$Q(3)=0$

abs. max: $(-3,0)(\frac{1}{4},0)(3,0)$

abs min: $(-1.8239,-1.38\times10^7)$

8. $h(w)=2w^3(w+2)^5$ on $[-\frac{5}{2},\frac{1}{2}]$ See separate Sheet

#7 9. $f(z)=\dfrac{z+4}{2z^2+z+8}$ on $[-10,0]$ $z=-4-3\sqrt{2}\approx-8.2426$ $z=-4+3\sqrt{2}\approx 0.2426$

$f(-10)=-\frac{1}{33}\approx-.0303$ $f(-8.2426)=-0.031328$ $f(0)=\frac{1}{2}$

abs. max: $(0,\frac{1}{2})$ abs min: $(-8.2426,-0.031328)$

10. $A(t)=t^2(10-t)^{\frac{2}{3}}$ on $[2,10.5]$ see separate Sheet

#12 11. $f(y)=\sin(\frac{y}{3})+\frac{2y}{9}$ on $[-10,15]$ $y=-6.9016$ $y=6.9016$ $y=11.9481$

$f(-10)=-2.0317$ $f(-6.9016)=-2.2790$ $f(6.9016)=2.2790$

$f(11.9481)=1.9098$ $f(15)=2.3744$

abs max: $(15,2.3744)$ abs. min $(-6.9016,-2.2790)$

#15 12. $g(w)=e^{w^3-2w^2-7w}$ on $[-\frac{1}{2},\frac{5}{2}]$ $w=\frac{7}{3}$

$g(-\frac{1}{2})=e^{\frac{23}{8}}$ $g(\frac{7}{3})=e^{-392/27}$ $g(\frac{5}{2})=e^{-115/8}$

abs. max: $(-\frac{1}{2},e^{23/8})$ abs. min: $(\frac{7}{3},e^{-392/27})$

#16 13. $R(x)=\ln(x^2+4x+14)$ on $[-4,2]$ $x=-2$

$R(-4)=2.6391$ $R(-2)=2.3026$ $R(2)=3.2581$

abs max: $(2,3.2581)$

abs. min: $(-2,2.3026)$

Finding Absolute Extrema

6. $g'(x) = 12x^3 - 78x^2 + 120x$ $\quad x=0,\ x=5/2,\ x=4$

$= 6x(x-4)(2x-5) = 0$

In interval

$x = 5/2,\ x = 4$

$g(1) = 26 \quad g(\frac{5}{2}) = 74.9375 \quad g(4) = 53 \quad g(5) = 114$

abs.max: $(5, 114)$ abs. min $(1, 26)$

8. $h'(w) = 6w^2(w+2)^5 + 2w^3 \cdot 5(w+2)^4$ $\quad x=0,\ x=-2,\ w=-3/4$

$= 2w^2(w+2)^4\,[3(w+2)+5w]$

$= 2w^2(w+2)^4(8w+6)$ $\quad h(-3/4) = -2.5749$

In interval

$w=-2$

$w=0$

$w=-3/4$

$h(-\frac{5}{2}) = 0.9766 \quad h(-2) = 0 \downarrow h(0) = 0 \quad h(1/2) = 24.4141$

abs max: $(1/2, 24.4141)$

abs. min: $(-3/4, -2.5749)$

10. $A'(t) = 2t(10-t)^{2/3} + t^2 \cdot \frac{2}{3}(10-t)^{-1/3} \cdot -1$ $\quad t=10,\ t=\frac{15}{2},\ t=0$

$$= 2t(10-t)^{2/3} - \frac{2t^2}{3(10-t)^{1/3}}$$

$$= \frac{6t(10-t) - 2t^2}{3(10-t)^{1/3}}$$

$$= \frac{60t - 6t^2 - 2t^2}{3(10-t)^{1/3}}$$

$$= \frac{-8t^2 + 60t}{3(10-t)^{1/3}} = \frac{4t(-2t+15)}{3(10-t)^{1/3}}$$

In interval

$t = \frac{15}{2}$

$t = 10$

$A(2) = 16 \quad A(\frac{15}{2}) = 103.613 \quad A(10) = 0 \quad A(10.5) = 69.4531$

abs max: $(\frac{15}{2}, 103.613)$

abs min: $(10, 0)$

The Shape of a Graph, Part I

5. $f'(x) = 6x^2 - 18x - 60$

$= 6(x^2 - 3x - 10)$

$= 6(x-5)(x+2)$

Critical points

$x = 5$

$x = -2$

Increasing $(-\infty, -2)$ and $(5, \infty)$

Decreasing $(-2, 5)$

6. $h'(t) = 120t^2 - 20t^3 - 20t^4$

$= -20t^2(t^2 + t - 6)$

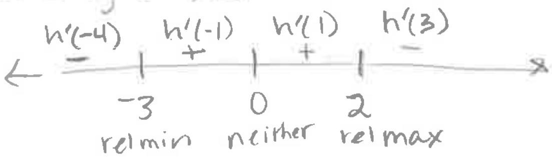
$= -20t^2(t+3)(t-2)$

Critical points

$t = -3$

$t = 2$

$t = 0$



Increasing $(-3, 0)$ and $(0, 2)$

Decreasing $(-\infty, -3)$ and $(2, \infty)$

7. $y' = 6x^2 - 20x + 12$

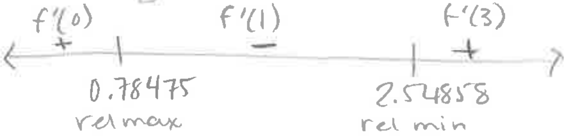
$= 2(3x^2 - 10x + 6)$

Critical points

$x = \frac{5 \pm \sqrt{7}}{3}$

$x \approx 0.78475$

$x \approx 2.54858$



Increasing $(-\infty, \frac{5-\sqrt{7}}{3})$ and $(\frac{5+\sqrt{7}}{3}, \infty)$

Decreasing $(\frac{5-\sqrt{7}}{3}, \frac{5+\sqrt{7}}{3})$

8. $p'(x) = -3\sin(3x) + 2$

$\sin(3x) = 2/3$

$\sin^{-1}(2/3) = 3x$

$3x = .7297 + 2\pi n$

$3x = 2.4119 + 2\pi n$

$\Rightarrow$

$x = 0.2432 + \frac{2\pi n}{3}$

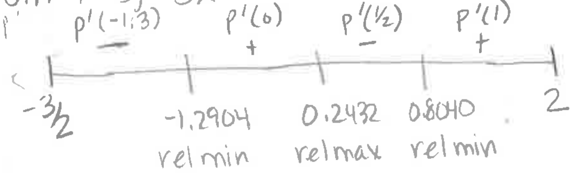
$x = 0.8040 + \frac{2\pi n}{3}$

Critical points

$x = -1.2904$

$x = 0.2432$

$x = 0.8040$



Increasing $(-1.2904, 0.2432)$ and $(0.8040, 2)$

Decreasing $(-3/2, -1.2904)$ and $(0.2432, 0.8040)$

9. $R'(z) = -5 - 7\cos(\frac{z}{2})$

$\cos(\frac{z}{2}) = -\frac{5}{7}$

$\cos^{-1}(-\frac{5}{7}) = \frac{z}{2}$

$\frac{z}{2} = 2.3664 + 2\pi n$

$\frac{z}{2} = 3.9168 + 2\pi n$

$\Rightarrow$

$z = 4.7328 + 4\pi n$

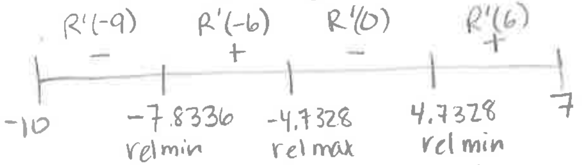
$z = 7.8336 + 4\pi n$

Critical points

$z = -7.8336$

$z = -4.7328$

$z = 4.7328$



Increasing $(-7.8336, -4.7328)$ and $(4.7328, 7)$

Decreasing $(-10, -7.8336)$ and $(-4.7328, 4.7328)$

10. $h'(t) = 2t(t-7)^{1/3} + t^2(t-7)^{-2/3} \cdot \frac{1}{3}$

$= 2t(t-7)^{1/3} + \frac{t^2}{3(t-7)^{2/3}}$

$= \frac{6t(t-7) + t^2}{(t-7)^{2/3}}$

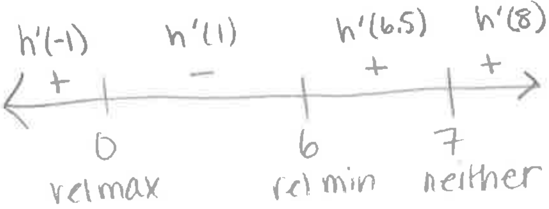
$= \frac{7t^2 - 42t}{(t-7)^{2/3}} = \frac{7t(t-6)}{(t-7)^{2/3}}$

Critical points

$t = 0$

$t = 6$

$t = 7$



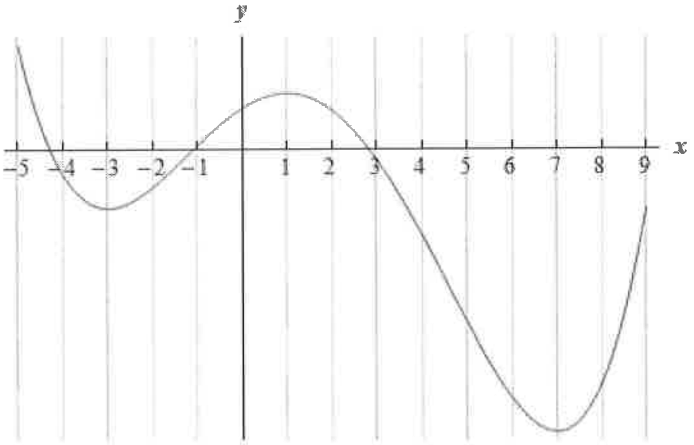
Increasing $(-\infty, 0), (6, 7), (7, \infty)$

Decreasing $(0, 6)$

The Shape of a Graph, Part I

For problems 1 & 2 the graph of a function is given. Determine the open intervals on which the function increases and decreases.

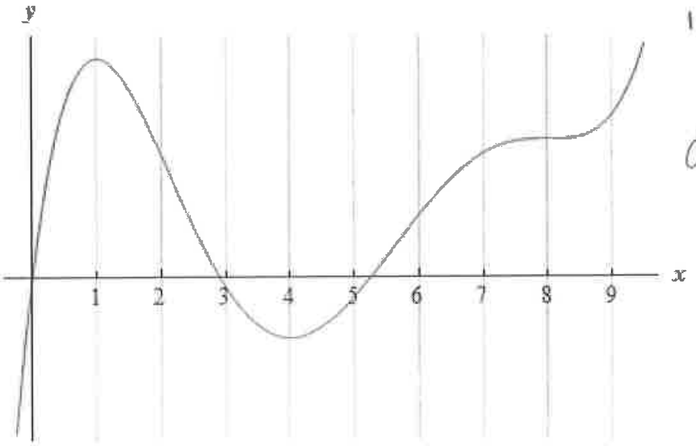
1.



Increasing:
(-3,1) and (7,∞)

Decreasing:
(-∞,-3) and (1, 7)

2.



increasing:
(-∞,1) and (4,8) and (8,∞)

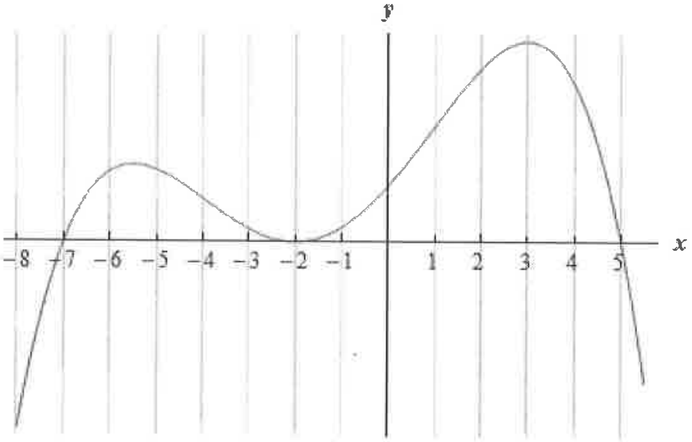
decreasing:
(1,4)

3. Below is the graph of the **derivative** of a function. From this graph determine the open intervals in which the **function** increases and decreases.

Derivative + increasing Derivative – decreasing

Increasing:
(-7,-2), (-2,5)

Decreasing:
(-∞,-7), (5,∞)



4. This problem is about some function. All we know about the function is that it exists everywhere and we also know the information given below about the derivative of the function. Answer each of the following questions about this function.

(a) Identify the critical points of the function.

(b) Determine the open intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

critical points $\longrightarrow$ $\overset{\text{rel max}}{f'(-5)=0}$ $\quad$ $\overset{\text{neither}}{f'(-2)=0}$ $\quad$ $\overset{\text{rel min}}{f'(4)=0}$ $\quad$ $\overset{\text{rel max}}{f'(8)=0}$

$f'(x)<0$ on $\underbrace{(-5,-2),\ (-2,4),\ (8,\infty)}_{\text{decreasing}}$ $\qquad$ $f'(x)>0$ on $\underbrace{(-\infty,-5),\ (4,8)}_{\text{increasing}}$

For problems 5 – 12 answer each of the following.

(a) Identify the critical points of the function.

(b) Determine the open intervals on which the function increases and decreases.

(c) Classify the critical points as relative maximums, relative minimums or neither.

5. $f(x)=2x^3-9x^2-60x$

6. $h(t)=50+40t^3-5t^4-4t^5$

See separate sheet

7. $y=2x^3-10x^2+12x-12$

8. $p(x)=\cos(3x)+2x$ on $\left[-\frac{3}{2},\,2\right]$

9. $R(z)=2-5z-14\sin\left(\frac{z}{2}\right)$ on $[-10,7]$

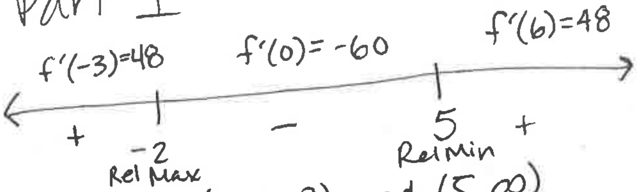
10. $h(t)=t^2\sqrt[3]{t-7}$

The Shape of a Graph, Part I

5. $f'(x)=6x^2-18x-60=0$

$6(x-5)(x+2)=0$

$x=5 \quad x=-2$



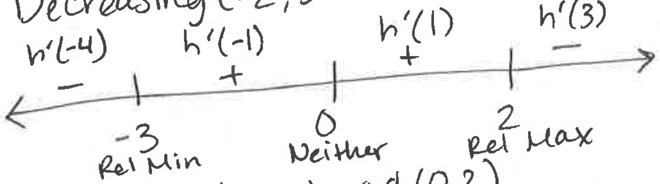
Increasing $(-\infty,-2)$ and $(5,\infty)$

Decreasing $(-2,5)$

6. $h'(t)=120t^2-20t^3-20t^4$

$=-20t^2(t+3)(t-2)=0$

$t=0 \quad t=-3 \quad t=2$



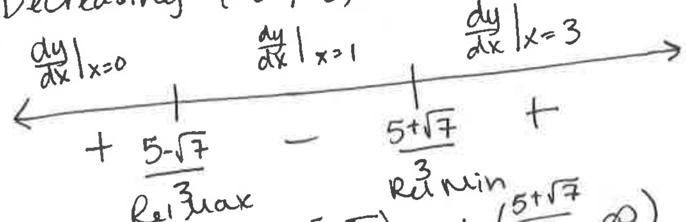
Increasing $(-3,0)$ and $(0,2)$

Decreasing $(-\infty,-3)$ and $(2,\infty)$

7. $y'=6x^2-20x+12$

$=2(3x^2-10x+6)=0$

quadratic $x=\frac{5\pm\sqrt{7}}{3}$



Increasing $(-\infty,\frac{5-\sqrt{7}}{3})$ and $(\frac{5+\sqrt{7}}{3},\infty)$

Decreasing $(\frac{5-\sqrt{7}}{3},\frac{5+\sqrt{7}}{3})$

8. $p'(x)=-\sin(3x)\cdot 3+2$

$=-3\sin(3x)+2=0$

$\sin(3x)=\frac{2}{3}$

$3x=.7297+2\pi n$

$3x=2.412+2\pi n$

$x=0.2432+\frac{2\pi n}{3}$

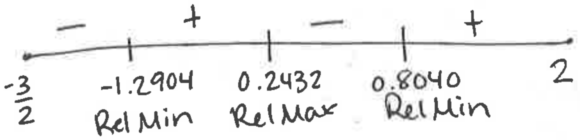
$x=0.8040+\frac{2\pi n}{3}$

In interval

$x=-1.2904$

$x=0.2432$

$x=0.8040$



Increasing: $(-1.2904,0.2432)$ and $(0.8040,2)$

Decreasing: $(-\frac{3}{2},-1.2904)$ and $(0.2432,0.8040)$

9. $R'(z)=-5-7\cos(\frac{z}{2})$

$\cos(\frac{z}{2})=-\frac{5}{7}$

$\frac{z}{2}=2.3664+2\pi n$

$\frac{z}{2}=3.9168+2\pi n$

$z=4.7328+4\pi n$

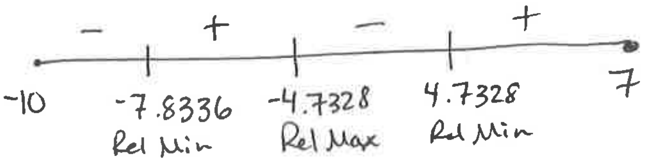
$z=7.8336+4\pi n$

In interval

$z=-7.8336$

$z=-4.7328$

$z=4.7328$



Increasing $(-7.8336,-4.7328)$ and $(4.7328,7)$

Decreasing $(-10,-7.8336)$ and $(-4.7328,4.7328)$

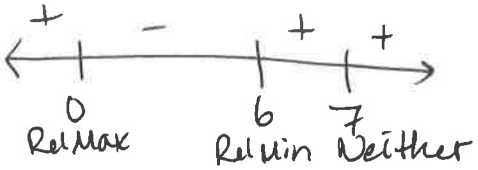
10. $n'(t) = 2t\sqrt[3]{t-7} + \frac{1}{3}t^2(t-7)^{-2/3}$

Common denom. $\frac{6t(t-7)}{3(t-7)^{2/3}} + \frac{t^2}{3(t-7)^{2/3}}$

$$\frac{7t^2-42t}{3(t-7)^{2/3}} = \frac{7t(t-6)}{3(t-7)^{2/3}}$$

Critical points

$t=0$
$t=6$
$t=7$



Increasing $(-\infty,0)$ and $(6,7)$ and $(7,\infty)$

Decreasing $(0,6)$

11. $f'(w) = e^{2-\frac{1}{2}w^2} + we^{2-\frac{1}{2}w^2}\cdot -w$

$= e^{2-\frac{1}{2}w^2} - w^2e^{2-\frac{1}{2}w^2}$

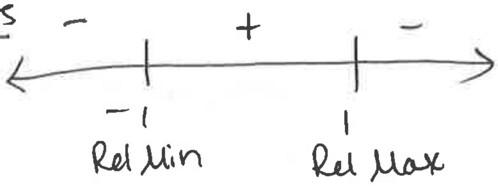
$= e^{2-\frac{1}{2}w^2}(1-w^2)$

$= e^{2-\frac{1}{2}w^2}(1-w)(1+w)$

↑ never zero

Critical points

$w=1$
$w=-1$



Increases $(-1,1)$

Decreases $(-\infty,-1)$ and $(1,\infty)$

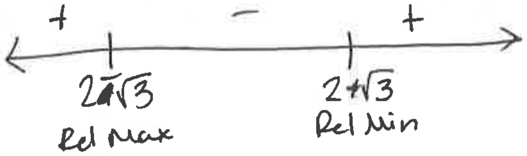
12. $g'(x) = 1 - 2\frac{1}{1+x^2}\cdot 2x$

$= 1 - \frac{4x}{1+x^2}$

common denom $= \frac{1+x^2-4x}{1+x^2}$ ← quadratic

Critical pts

$x = 2\pm\sqrt{3}$



Increasing $(-\infty, 2-\sqrt{3})$ and $(2+\sqrt{3},\infty)$

Decreasing $(2-\sqrt{3}, 2+\sqrt{3})$

13. a. Rel Max at $x=4$ $f'(x) = 4-x$

Critical point at $x=4$

Increasing before, decreasing after

b. Since the derivative brings the power down and decreases it by 1: $f(x) = 4x - \frac{1}{2}x^2 + C$

Some constant "disappears" when you take the derivative

c. If $f(x) = 4x - \frac{1}{2}x^2 + c$ and $f(4)=1$ $1 = 4(4) - \frac{1}{2}(4)^2 + c$ so, $f(x) = 4x - \frac{1}{2}x^2 - 7$

$1 = 8 + c$

$c = -7$

11. $f(w) = w\mathbf{e}^{2-\frac{1}{2}w^2}$

See separate sheet

12. $g(x) = x - 2\ln(1 + x^2)$

13. For some function, $f(x)$, it is known that there is a relative maximum at $x = 4$. Answer each of the following questions about this function.

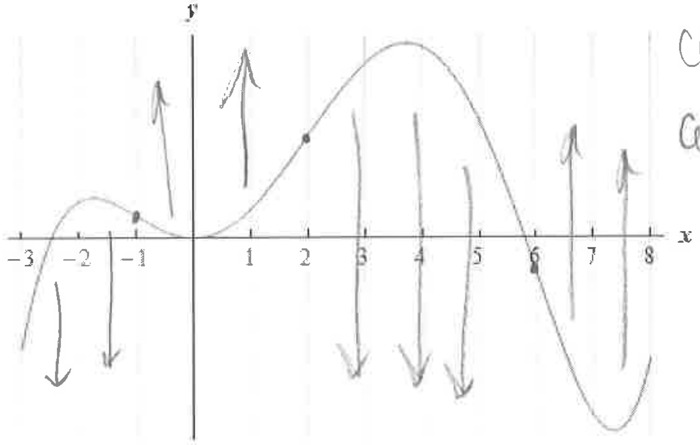
- **(a)** What is the simplest form for the derivative of this function? Note : There really are many possible forms of the derivative so to make the rest of this problem as simple as possible you will want to use the simplest form of the derivative that you can come up with.
- **(b)** Using your answer from **(a)** determine the most general form of the function.
- **(c)** Given that $f(4) = 1$ find a function that will have a relative maximum at $x = 4$. Note : You should be able to use your answer from **(b)** to determine an answer to this part.

~~14~~. Given that $f(x)$ and $g(x)$ are increasing functions. If we define $h(x) = f(x) + g(x)$ show that $h(x)$ is an increasing function.

~~15~~. Given that $f(x)$ is an increasing function and define $h(x) = [f(x)]^2$. Will $h(x)$ be an increasing function? If yes, prove that $h(x)$ is an increasing function. If not, can you determine any other conditions needed on the function $f(x)$ that will guarantee that $h(x)$ will also increase?

The Shape of a Graph, Part II

1. The graph of a function is given below. Determine the open intervals on which the function is concave up and concave down.



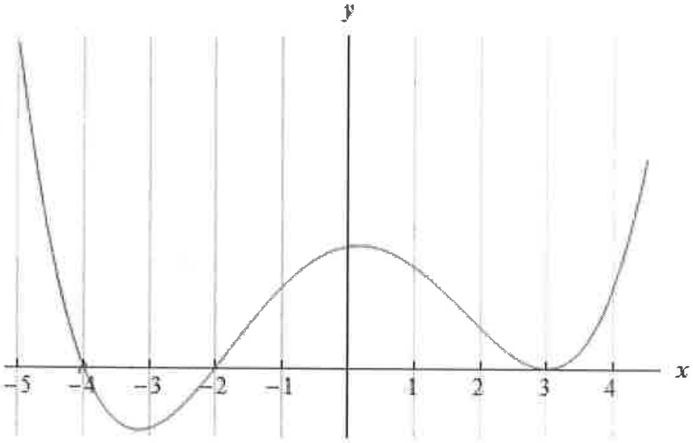
Concave up: (-1,2) & (6,∞)

Concave down: (-∞,-1) & (2,6)

2. Below is the graph the 2nd **derivative** of a function. From this graph determine the open intervals in which the **function** is concave up and concave down.

Concave up:
(-∞,-4), (-2,3), (3,∞)

Concave down
(-4,-2)



For problems 3 – 8 answer each of the following.

(a) Determine a list of possible inflection points for the function.
(b) Determine the open intervals on which the function is concave up and concave down.
(c) Determine the inflection points of the function.

3. $f(x) = 12 + 6x^2 - x^3$ $f'(x) = 12x - 3x^2$ $f''(x) = 12 - 6x$ $\boxed{x=2}$ inflection point

concave up (-∞,2) down(2,∞)

4. $g(z) = z^4 - 12z^3 + 84z + 4$ $g'(z) = 4z^3 - 36z^2 + 84$ $g''(z) = 12z^2 - 72z$ $\boxed{z=0,\ z=6}$

concave up (-∞,0) & (6,∞) down (0,6)

5. $h(t) = t^4 + 12t^3 + 6t^2 - 36t + 2$

6. $h(w) = 8 - 5w + 2w^2 - \cos(3w)$ on $[-1, 2]$

7. $R(z) = z(z+4)^{\frac{2}{3}}$

8. $h(x) = \mathbf{e}^{4-x^2}$

For problems 9 – 14 answer each of the following.

(a) Identify the critical points of the function.
(b) Determine the open intervals on which the function increases and decreases.
(c) Classify the critical points as relative maximums, relative minimums or neither.
(d) Determine the open intervals on which the function is concave up and concave down.

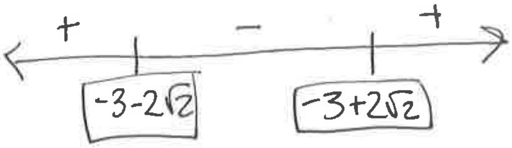
Shape of a Graph, Part II

5. $h'(t) = 4t^3 + 36t^2 + 12t - 36$

$h''(t) = 12t^2 + 72t + 12 = 0$

$12(t^2 + 6t + 1) = 0$

quadratic $t = -3 \pm 2\sqrt{2}$



Concave up: $(-\infty, -3-2\sqrt{2})$ & $(3+2\sqrt{2}, \infty)$

Concave down: $(-3-2\sqrt{2}, -3+2\sqrt{2})$

6. $h'(w) = -5 + 4w + 3\sin(3w)$

$h''(w) = 4 + 9\cos(3w) = 0$

$\cos(3w) = -4/9$

$3w = 2.0314 + 2\pi n$

$3w = 4.2518 + 2\pi n$

$w = 0.6771 + \frac{2\pi n}{3}$

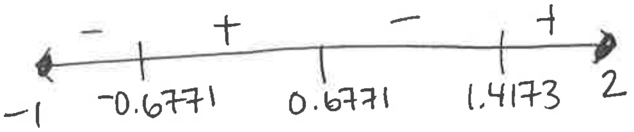
$w = 1.4173 + \frac{2\pi n}{3}$

Inflection Pts:

$w = -0.6771$

$w = 0.6771$

$w = 1.4173$



Concave up: $(-0.6771, 0.6771)$ & $(1.4173, 2)$

Concave down: $(-1, -0.6771)$ & $(0.6771, 1.4173$

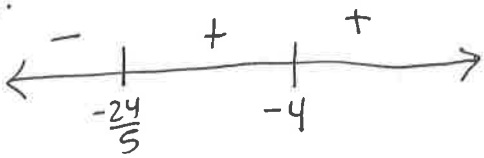
7. $R'(z) = (z+4)^{2/3} + \frac{2}{3}z(z+4)^{-1/3}$

Common denom. $\frac{3(z+4)}{3(z+4)^{1/3}} + \frac{2z}{3(z+4)^{1/3}}$

Inflection pts:

~~$z = -4$~~

$z = -\frac{24}{5}$



$R'(z) = \frac{5z+12}{3(z+4)^{1/3}}$

Concave up: $(-\frac{24}{5}, -4)$ and $(-4, \infty)$

Concave down: $(-\infty, -24/5)$

$$R''(z) = \frac{5(3(z+4)^{1/3}) - (5z+12)(z+4)^{-2/3}}{[3(z+4)^{1/3}]^2}$$

Factor out a $(z+4)^{-2/3}$ ... subtract exponents $(z+4)^{1/3 - -2/3} = (z+4)^{1}$

$$= \frac{(z+4)^{-2/3}[15(z+4) - (5z+12)]}{[3(z+4)^{1/3}]^2}$$

$$= \frac{(10z+48)}{9(z+4)^{2/3}(z+4)^{2/3}} = \frac{10z+48}{9(z+4)^{4/3}}$$

8. $h'(x) = -2xe^{4-x^2}$

$h''(x) = -2e^{4-x^2} - 2xe^{4-x^2} \cdot (-2x)$

$= \underset{\substack{\uparrow \\ \text{never zero}}}{-2e^{-4-x^2}}(1-2x^2) = 0$

$1 - 2x^2 = 0$

$x^2 = \frac{1}{2}$

$x = \pm\frac{\sqrt{2}}{2}$

Inflection Pts:

$x = \frac{\sqrt{2}}{2}$

$x = -\frac{\sqrt{2}}{2}$

Concave up $(-\infty, \frac{-\sqrt{2}}{2}), (\frac{\sqrt{2}}{2}, \infty)$

Concave down $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

~~Homework: due ____________________~~

~~pages 251 – 254 #17, 20, 25, 26, 29, 30,~~
~~31, 33~~

(e) Determine the inflection points of the function.
(f) Use the information from steps **(a)** – **(e)** to sketch the graph of the function.

9. $g(t) = t^5 - 5t^4 + 8$

10. $f(x) = 5 - 8x^3 - x^4$

11. $h(z) = z^4 - 2z^3 - 12z^2$

12. $Q(t) = 3t - 8\sin\left(\frac{t}{2}\right)$ on $[-7, 4]$

13. $f(x) = x^{\frac{4}{3}}(x-2)$

14. $P(w) = w\mathbf{e}^{4w}$ on $\left[-2, \frac{1}{4}\right]$

15. Determine the minimum degree of a polynomial that has exactly one inflection point.

16. Suppose that we know that $f(x)$ is a polynomial with critical points $x = -1$, $x = 2$ and $x = 6$. If we also know that the 2nd derivative is $f''(x) = 3x^2 + 14x - 4$. If possible, classify each of the critical points as relative minimums, relative maximums. If it is not possible to classify the critical points clearly explain why they cannot be classified.

The Mean Value Theorem

For problems 1 & 2 determine all the number(s) c which satisfy the conclusion of Rolle's Theorem for the given function and interval.

1. $f(x) = x^2 - 2x - 8$ on $[-1, 3]$

2. $g(t) = 2t - t^2 - t^3$ on $[-2, 1]$

For problems 3 & 4 determine all the number(s) c which satisfy the conclusion of the Mean Value Theorem for the given function and interval.

3. $h(z) = 4z^3 - 8z^2 + 7z - 2$ on $[2, 5]$

4. $A(t) = 8t + \mathbf{e}^{-3t}$ on $[-2,3]$

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7,0]$, that $f(-7) = -3$ and that $f'(x) \le 2$. What is the largest possible value for $f(0)$?

6. Show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

Optimization

1. Find two positive numbers whose sum is 300 and whose product is a maximum.

2. Find two positive numbers whose product is 750 and for which the sum of one and 10 times the other is a minimum.

3. Let x and y be two positive numbers such that $x + 2y = 50$ and $(x+1)(y+2)$ is a maximum.

4. We are going to fence in a rectangular field. If we look at the field from above the cost of the vertical sides are \$10/ft, the cost of the bottom is \$2/ft and the cost of the top is \$7/ft. If we have \$700 determine the dimensions of the field that will maximize the enclosed area.

5. We have 45 m^2 of material to build a box with a square base and no top. Determine the dimensions of the box that will maximize the enclosed volume.

6. We want to build a box whose base length is 6 times the base width and the box will enclose 20 in^3. The cost of the material of the sides is \$3/in^2 and the cost of the top and bottom is \$15/in^2. Determine the dimensions of the box that will minimize the cost.

7. We want to construct a cylindrical can with a bottom but no top that will have a volume of 30 cm^3. Determine the dimensions of the can that will minimize the amount of material needed to construct the can.

8. We have a piece of cardboard that is 50 cm by 20 cm and we are going to cut out the corners and fold up the sides to form a box. Determine the height of the box that will give a maximum volume.

More Optimization Problems

Optimization

1. x = first # y = second #

Constraint

$x+y=300$

$x=300-y$

Maximize

$P=xy$

$P=(300-y)y$

$P=300y-y^2$

$P'=300-2y=0$

$300=2y$

$\boxed{y=150 \quad x=150}$

2. x = pos # y = 2nd pos #

Constraint

$xy=750$

$x=\frac{750}{y}$

Minimum

$S=x+10y$

$S=\frac{750}{y}+10y$

$S'=-\frac{750}{y^2}+10=0$

$-\frac{750}{y^2}=-10$

$-750=-10y^2$

$75=y^2$

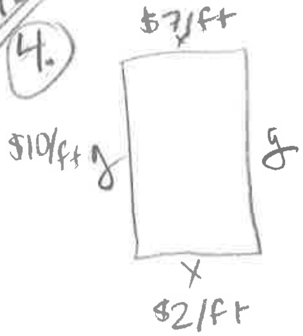
$\boxed{y=5\sqrt{3} \quad x=50\sqrt{3}}$

$x\cdot 5\sqrt{3}=750$

$x=\frac{150}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$

Oops! be careful!

④.



Constraint:

$10y(2)+7x+2x=700$

$20y+9x=700$

$9x=700-20y$

$x=\frac{700}{9}-\frac{20}{9}y$

Maximize

$A=xy$

$A=\left(\frac{700}{9}-\frac{20}{9}y\right)y$

$A=\frac{700}{9}y-\frac{20}{9}y^2$

$A'=\frac{700}{9}-\frac{40}{9}y=0$

$\frac{700}{9}=\frac{40}{9}y$

$y=\frac{700}{40}=\frac{35}{2}$

$x=\frac{700}{9}-\frac{20}{9}\left(\frac{35}{2}\right)=\frac{350}{9}$

$\boxed{x=\frac{350}{9} \quad y=\frac{35}{2}}$

③. Constraint

$x+2y=50$

$x=50-2y$

Maximize

$(x+1)(y+2)=P$

$(50-2y+1)(y+2)=P$

$(51-2y)(y+2)=P$

$51y-2y^2+102-4y=P$

$-2y^2+47y+102=P$

$P'=-4y+47=0$

$-4y=-47$

$y=\frac{47}{4}$

$x+2\left(\frac{47}{4}\right)=50$

$\boxed{x=\frac{53}{2} \quad y=\frac{47}{4}}$

5.

Constraint

$4xh+x^2=45$

$4xh=45-x^2$

$h=\frac{45-x^2}{4x}$

$\boxed{x=\sqrt{15} \quad h=\frac{\sqrt{15}}{2}}$

Maximize

$V=x^2h$

$V=x^2\left(\frac{45-x^2}{4x}\right)$

$V=\frac{x}{4}(45-x^2)$

$V=\frac{45x}{4}-\frac{x^3}{4}$

$V'=\frac{45}{4}-\frac{3}{4}x^2=0$

$\frac{45}{4}=\frac{3}{4}x^2$

$45=3x^2$

$15=x^2$

$x=\sqrt{15}$

$4(\sqrt{15})h+(\sqrt{15})^2=45$

$4\sqrt{15}h+15=45$

$4\sqrt{15}h=30$

$h=\frac{15}{2\sqrt{15}}\cdot\frac{\sqrt{15}}{\sqrt{15}}=\frac{\sqrt{15}}{2}$

6.

$V=20$

Constraint

$6w^2h=20$

$h=\frac{10}{3w^2}$

Minimize

Cost

$C=2(6wh)\cdot\$3+2(wh)\cdot\$3+2(6w^2)\cdot\$15$

$C=36wh+6wh+180w^2$

$C=42wh+180w^2$

$C=42w\left(\frac{10}{3w^2}\right)+180w^2$

$C=\frac{140}{w}+180w^2$

$C'=\frac{-140}{w^2}+360w=0$

$\frac{-140}{w^2}=-360w$

$+360w^3=+140$

$w^3=\frac{7}{18}$

$w=\sqrt[3]{\frac{7}{18}}\approx 0.7299$

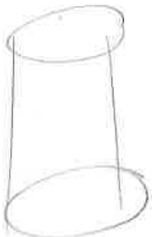
$$\boxed{\begin{array}{l} w=\sqrt[3]{\frac{7}{18}}\approx 0.7299 \\ h=6.256 \\ l=4.3794 \end{array}}$$

$6\left(\sqrt[3]{\frac{7}{18}}\right)^2h=20$

use decimals here

$6\cdot 0.7299$

7.



Constraint

$V=30$

$\pi r^2h=30$

$h=\frac{30}{\pi r^2}$

Minimize

$SA=\pi r^2+2\pi rh$

$SA=\pi r^2+2\cancel{\pi}\cancel{r}\left(\frac{30}{\cancel{\pi} r^{\cancel{2}}}\right)$

$SA=\pi r^2+60r^{-1}$

$SA'=2\pi r-60r^{-2}$

$2\pi r=\frac{60}{r^2}$

$2\pi r^3=60$

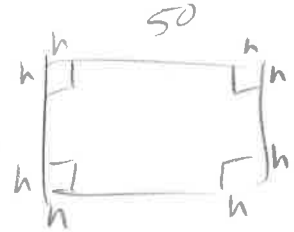
$\pi r^3=30$

$r=\sqrt[3]{\frac{30}{\pi}}\approx 2.1216$

$$\boxed{\begin{array}{l} r\approx 2.1216 \\ h\approx 2.1215 \end{array}}$$

$\pi(2.1216)^2h=30$

8.



Maximize Volume

$V=lwh$

$V=(50-2h)(20-2h)h$

$V=(1000-140h+4h^2)h$

$V=1000h-140h^2+4h^3$

$V'=1000-280h+12h^2=0$

Quadratic $h=\frac{35\pm5\sqrt{19}}{3}\approx 4.4018$ or ~~18.9315~~

$$\boxed{h=4.4018}$$

~~1. We want to construct a window whose middle is a rectangle and the top and bottom of the window are semi-circles. If we have 50 meters of framing material what are the dimensions of the window that will let in the most light?~~

~~2. Determine the area of the largest rectangle that can be inscribed in a circle of radius 1.~~

~~3. Find the point(s) on $x = 3 - 2y^2$ that are closest to $(-4, 0)$.~~

~~4. An 80 cm piece of wire is cut into two pieces. One piece is bent into an equilateral triangle and the other will be bent into a rectangle with one side 4 times the length of the other side. Determine where, if anywhere, the wire should be cut to maximize the area enclosed by the two figures.~~

~~5. A line through the point $(2, 5)$ forms a right triangle with the x-axis and y-axis in the 1st quadrant. Determine the equation of the line that will minimize the area of this triangle.~~

~~6. A piece of pipe is being carried down a hallway that is 18 feet wide. At the end of the hallway there is a right-angled turn and the hallway narrows down to 12 feet wide. What is the longest pipe (always keeping it horizontal) that can be carried around the turn in the hallway?~~

~~7. Two 10 meter tall poles are 30 meters apart. A length of wire is attached to the top of each pole and it is staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?~~

Indeterminate Forms and L'Hospital's Rule

Use L'Hospital's Rule to evaluate each of the following limits.

1. $\lim\limits_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \frac{0}{0}$ $\quad \lim\limits_{x \to 2} \frac{3x^2 - 14x + 10}{2x + 1} = \boxed{\frac{-6}{5}}$

2. $\lim\limits_{w \to -4} \frac{\sin(\pi w)}{w^2 - 16} = \frac{0}{0}$ $\quad \lim\limits_{w \to -4} \frac{\overset{\text{chain}}{\pi} \cos(\pi w)}{2w} = \boxed{\frac{\pi}{-8}}$

3. $\lim\limits_{t \to \infty} \frac{\ln(3t)}{t^2} = \frac{\infty}{\infty}$ $\quad \lim\limits_{t \to \infty} \frac{\frac{3}{3t}}{2t} = \frac{\frac{1}{t}}{2t} = \frac{1}{2t^2} = \boxed{0}$

4. $\lim\limits_{z \to 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2} = \frac{0}{0}$ $\quad \lim\limits_{z \to 0} \frac{2\cos(2z) + 14z - 2}{4z^3 + 6z^2 + 2z} = \frac{0}{0}$ $\quad \lim\limits_{z \to 0} \frac{-4\sin(2z) + 14}{12z^2 + 12z + 2} = \frac{14}{2} = \boxed{7}$

expand so you don't have to use product rule

$z^4 + 2z^3 + z^2$

5. $\lim\limits_{x\to-\infty}\dfrac{x^2}{\mathbf{e}^{1-x}}$ $= \dfrac{\infty}{\infty}$ $\lim\limits_{x\to-\infty}\dfrac{2x}{-e^{1-x}} = \dfrac{-\infty}{-\infty}$ $\lim\limits_{x\to-\infty}\dfrac{2}{e^{1-x}} = \boxed{0}$

6. $\lim\limits_{z\to\infty}\dfrac{z^2+\mathbf{e}^{4z}}{2z-\mathbf{e}^{z}}$ $= \dfrac{\infty}{-\infty}$ $\lim\limits_{z\to\infty}\dfrac{2z+4e^{4z}}{2-e^{z}} = \dfrac{\infty}{-\infty}$ $\lim\limits_{z\to\infty}\dfrac{2+16e^{4z}}{-e^{z}} = (2+16e^{4z})(-e^{-z}) = -2e^{-z}-16e^{3z}$

plugin ∞ $\uparrow$ $0-\infty=\boxed{-\infty}$

7. $\lim\limits_{t\to\infty}\left[t\ln\left(1+\dfrac{3}{t}\right)\right]$ $\lim\limits_{t\to\infty}\dfrac{\ln(1+\frac{3}{t})}{\frac{1}{t}} = \dfrac{0}{0}$ $\lim\limits_{t\to\infty}$

y move down

8. $\lim\limits_{w\to0^+}\left[w^2\ln\left(4w^2\right)\right]$

9. $\lim\limits_{x\to1^+}\left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right]$

10. $\lim\limits_{y\to0^+}\left[\cos(2y)\right]^{1/y^2}$

11. $\lim\limits_{x\to\infty}\left[\mathbf{e}^{x}+x\right]^{1/x}$

Linear Approximations

For problems 1 & 2 find a linear approximation to the function at the given point.

1. $f(x)=3x\mathbf{e}^{2x-10}$ at $x=5$

2. $h(t)=t^4-6t^3+3t-7$ at $t=-3$

3. Find the linear approximation to $g(z)=\sqrt[4]{z}$ at $z=2$. Use the linear approximation to approximate the value of $\sqrt[4]{3}$ and $\sqrt[4]{10}$. Compare the approximated values to the exact values.

4. Find the linear approximation to $f(t)=\cos(2t)$ at $t=\frac{1}{2}$. Use the linear approximation to approximate the value of $\cos(1)$ and $\cos(9)$. Compare the approximated values to the exact values.

5. Without using any kind of computational aid use a linear approximation to estimate the value of $\mathbf{e}^{0.1}$.

Indeterminate Forms & L'Hôpital's Rule

7. $\ln(1+\frac{3}{t}) = \ln(1+3t^{-1})$

$$\lim_{t\to\infty} \frac{\frac{-3t^{-2}}{1+3t^{-1}}}{-t^{-2}} = \frac{-3t^{-2}}{-t^{-2}-3t^{-3}} = \frac{\cancel{-t^{-2}}(3)}{\cancel{-t^{-2}}(1+3t^{-1})} = \frac{3}{1+\underset{\text{zero}}{\frac{3}{t}}} = \frac{3}{1} = \boxed{3}$$

8. $$\lim_{w\to 0^+} \frac{\ln(4w^2)}{w^{-2}} = \frac{-\infty}{\infty} \qquad \lim_{w\to 0^+} \frac{\frac{8w}{4w^2}}{w^{-2}} = \frac{2w^{-1}}{-2w^{-3}} = -w^2 = \boxed{0}$$

9. $$\lim_{x\to 1^+} \frac{\tan(\frac{\pi}{2}x)}{(x-1)^{-1}} = \frac{\infty}{0} \qquad \lim_{x\to 1^+} \frac{\frac{\pi}{2}\sec^2(\frac{\pi}{2}x)}{-1(x-1)^{-2}}$$ Not helping!!

$$\lim_{x\to 1^+} \frac{x-1}{\frac{1}{\tan(\frac{\pi}{2}(x))}} = \frac{x-1}{\cot(\frac{\pi}{2}x)} = \frac{0}{0} \qquad \lim_{x\to 1^+} \frac{1}{-\frac{\pi}{2}\csc^2(\frac{\pi}{2}x)} = \frac{1}{-\frac{\pi}{2}} = \boxed{-\frac{2}{\pi}}$$

10. $$\lim_{y\to 0^+} \cos(2y)^{1/y^2} = 1^\infty$$

$z = \cos(2y)^{1/y^2}$

$\ln z = \ln \cos(2y)^{1/y^2}$

$\ln z = \frac{1}{y^2} \ln \cos(2y)$

$$\overset{[\ln z]}{\lim_{y\to 0^+}} \frac{\ln \cos(2y)}{y^2} = \frac{0}{0} \qquad \overset{[\ln z]}{\lim_{y\to 0^+}} \frac{\frac{-2\sin(2y)}{\cos(2y)}}{2y} = \frac{-\tan(2y)}{y} = \frac{0}{0}$$

$$\overset{[\ln z]}{\lim_{y\to 0^+}} \frac{-2\sec^2(2y)}{1} = -2$$ so: $\ln z = -2$ ∴ $z = \boxed{e^{-2}}$

11. $$\lim_{x\to\infty} [e^x + x]^{1/x}$$

$z = (e^x + x)^{1/x}$

$\ln z = \frac{1}{x} \ln(e^x + x)$

$\ln z = \frac{\ln(e^x+x)}{x}$

$$\overset{[\ln z]}{\lim_{x\to\infty}} \frac{\ln(e^x+x)}{x} = \frac{\infty}{\infty}$$

$$\overset{[\ln z]}{\lim_{x\to\infty}} \frac{\frac{e^x+1}{e^x+x}}{1} = \frac{e^x+1}{e^x+x} = \frac{\infty}{\infty}$$

$$\overset{[\ln z]}{\lim_{x\to\infty}} \frac{e^x}{e^x+1} = \frac{\infty}{\infty}$$

$$\overset{[\ln z]}{\lim_{x\to\infty}} \frac{e^x}{e^x} = 1$$ ∴ $\ln z = 1$ ∴ $z = \boxed{e}$

Differentials

For problems 1 – 3 compute the differential of the given function.

1. $f(x) = x^2 - \sec(x)$

2. $w = \mathbf{e}^{x^4 - x^2 + 4x}$

3. $h(z) = \ln(2z)\sin(2z)$

4. Compute *dy* and Δy for $y = \mathbf{e}^{x^2}$ as *x* changes from 3 to 3.01.

5. Compute *dy* and Δy for $y = x^5 - 2x^3 + 7x$ as *x* changes from 6 to 5.9.

6. The sides of a cube are found to be 6 feet in length with a possible error of no more than 1.5 inches. What is the maximum possible error in the volume of the cube if we use this value of the length of the side to compute the volume?

Newton's Method

For problems 1 & 2 use Newton's Method to determine x_2 for the given function and given value of x_0.

1. $f(x) = x^3 - 7x^2 + 8x - 3$, $x_0 = 5$

2. $f(x) = x\cos(x) - x^2$, $x_0 = 1$

For problems 3 & 4 use Newton's Method to find the root of the given equation, accurate to six decimal places, that lies in the given interval.

3. $x^4 - 5x^3 + 9x + 3 = 0$ in $[4,6]$

4. $2x^2 + 5 = \mathbf{e}^x$ in $[3,4]$

~~For problems 5 & 6 use Newton's Method to find all the roots of the given equation accurate to six decimal places.~~

~~5. $x^3 - x^2 - 15x + 1 = 0$~~

~~6. $2 - x^2 = \sin(x)$~~

Business Applications

1. A company can produce a maximum of 1500 widgets in a year. If they sell x widgets during the year then their profit, in dollars, is given by,

$$P(x) = 30{,}000{,}000 - 360{,}000x + 750x^2 - \tfrac{1}{3}x^3$$

How many widgets should they try to sell in order to maximize their profit?

2. A management company is going to build a new apartment complex. They know that if the complex contains x apartments the maintenance costs for the building, landscaping etc. will be,

$$C(x) = 4000 + 14x - 0.04x^2$$

The land they have purchased can hold a complex of at most 500 apartments. How many apartments should the complex have in order to minimize the maintenance costs?

3. The production costs, in dollars, per day of producing x widgets is given by,

$$C(x) = 1750 + 6x - 0.04x^2 + 0.0003x^3$$

What is the marginal cost when $x = 175$ and $x = 300$? What do your answers tell you about the production costs?

4. The production costs, in dollars, per month of producing x widgets is given by,

$$C(x) = 200 + 0.5x + \frac{10000}{x}$$

What is the marginal cost when $x = 200$ and $x = 500$? What do your answers tell you about the production costs?

5. The production costs, in dollars, per week of producing x widgets is given by,

Business Applications

1. $P(x) = 30,000,000 - 360,000x + 750x^2 - \frac{1}{3}x^3$

max: 1500/year
$P(x)$ = profit
x = # of widgets

$P'(x) = -360,000 + 1500x - x^2$
$= -(x^2 - 1500x + 36,000)$
$0 = -(x-1200)(x-300)$
$x = 1200, \quad x = 300$

Test points:

$x=0$
$P(0) = 30,000,000$

$x = 300$
$P(300) = -19,500,000$

$x=1200$
$P(1200) = 102,000,000$

$x=1500$
$P(1500) = 52,500,000$

the company should sell 1200 widgets to maximize profit.

2. $C(x) = 4000 + 14x - 0.04x^2$

$C(x)$ = cost of maintenance
x = # of apartments
Max: 500 apts

$C'(x) = 14 - 0.08x = 0$
$x = 175$

Test points:

$x=0$
$C(0) = 4000$

$x=175$
$C(175) = 5225$

$x=500$
$C(500) = 1000$

The complex should have 500 apartments to minimize maintenance costs.

3. $C(x) = 1750 + 6x - 0.04x^2 + 0.0003x^3$

$C(x)$ = production costs \$\$
x = widgets
marginal cost @ $x=175$
$x=300$

$C'(x) = 6 - 0.08x + 0.0009x^2$
$C'(175) = 6 - 0.08(175) + 0.0009(175)^2 = \$19.5625$
$C'(300) = 6 - 0.08(300) + 0.0009(300)^2 = \$63.00$

It will cost \$19.56 to produce the 176th widget and \$63.00 to produce the 301st widget.

4. $C(x) = 200 + 0.5x + \frac{10000}{x}$

$C(x)$ = production cost
x = widgets
Marginal cost @ $x = 200$
$x = 500$

$C'(x) = 0.5 - \frac{10000}{x^2}$

$C'(200) = 0.5 - \frac{10000}{(200)^2} = 0.25$

$C'(500) = 0.5 - \frac{10000}{(500)^2} = 0.46$

It will cost approximately \$0.25 for the 201st widget and \$0.46 for the 501st widget.

5. $C(x) = 4000 - 32x + 0.08x^2 + 0.00006x^3$

$P(x) = 250 + 0.02x - 0.001x^2$

$C(x)$ = production cost
x = widgets
$P(x)$ = demand function
Marginal cost, revenue, and profit at $x = 200$
$x = 400$

Marginal cost

$C'(x) = -32 + 0.16x + 0.00018x^2$

$C'(200) = -32 + 0.16(200) + 0.00018(200)^2 = \$7.20$

$C'(400) = -32 + 0.16(400) + 0.00018(400)^2 = \$60.80$

It will cost approx. \$7.20 to produce the 201st widget and \$60.80 to produce the 401st widget

Revenue

$R(x) = x\,p(x) = 250x + 0.02x^2 - 0.001x^3$

$R'(x) = 250 + 0.04x - 0.003x^2 = \$138$

$R'(200) = 250 + 0.04(200) - 0.003(200)^2 = -\$214$

$R'(400) = 250 + 0.04(400) - 0.003(400)^2$

The 201st widget will add \$138 in revenue, the 401st widget will decrease revenue by \$214.

Profit

$P(x) = R(x) - C(x) = \underbrace{250x + 0.02x^2 - 0.001x^3}^{R(x)}, + (-4000 + 32x \mp 0.08x^2 \mp 0.00006x^3)$

$P(x) = -4000 + 282x - 0.06x^2 - 0.00106x^3$

$P'(x) = 282 - 0.12x - 0.00318x^2 = \$130.80$

$P'(200) = 282 - 0.12(200) - 0.00318(200)^2 = -\$274.80$

$P'(400) = 282 - 0.12(400) - 0.00318(400)^2$

The 201st widget will increase profit by \$130.80 and the 401st widget will decrease profit by \$274.80